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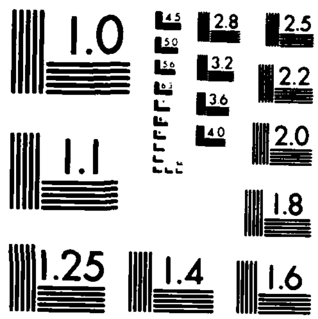

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APPLICATION OF MATCHED EXPANSION METHODS TO PROBLEMS IN ACOUSTICS

APPLICATION OF MATCHED EXPANSION METHODS TO PROBLEMS IN ACOUSTICS

by

D.G. Crighton

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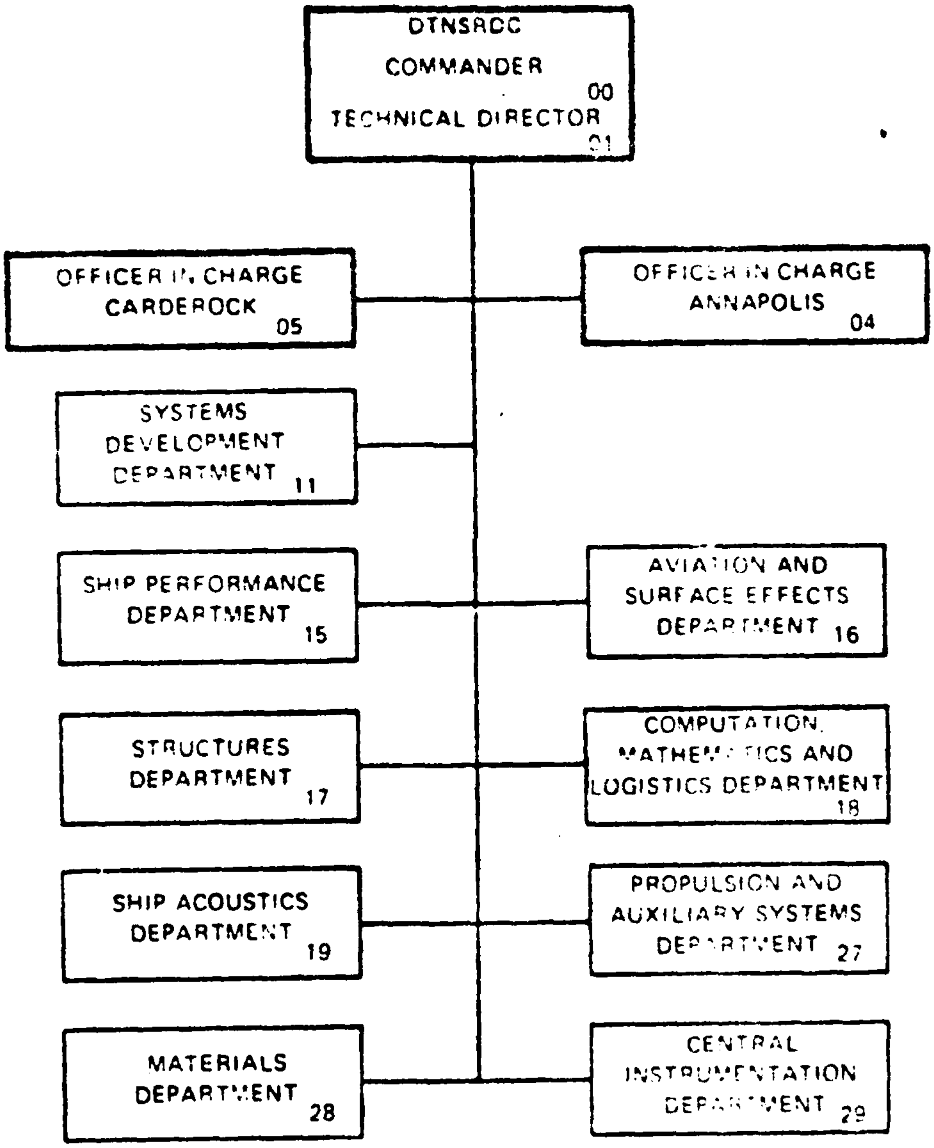

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Many problems in acoustics contain a small dimensionless parameter ϵ, and it is useful, both conceptually and from the point of view of numerical computation, to seek a solution in the form of a perturbation series. In the simplest cases the series would proceed by integral powers of ϵ,

$$\phi = \phi_o + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots.$$

where, in a typical application, ϕ might be an acoustical potential or pressure and ϵ a frequency parameter, or Helmholtz number. In *most* cases, however, the problem of determining the functions ϕ_i is a *singular*

20. (continued)

perturbation problem, that is, one in which no single series like the one quoted will be valid both in the near field (where boundary data are specified and where surface loading may be of interest) and in the far field (where the signal directivity and level are required). Separate series must be developed describing the near and far fields, but neither can be completely constructed independently of the other because each series lacks sufficient boundary data for its unique determination.

This report describes how the method of Matched Asymptotic Expansions (MAE) can be used safely and systematically (1) to indicate the appropriate form taken by the inner (near field) and outer (wave field) series and (2) to determine all unknown functions and constants appearing in both series by "matching" the series according to a clear-cut rule. These points are illustrated by detailed study of several very simple problems in low-frequency acoustic scattering problems which serve to demonstrate that physical arguments are unreliable in these problems and that they are no substitute for the unambiguous matching rule. Two-dimensional scattering problems are used to introduce logarithmic gauge functions; it is shown that the matching rule can easily accommodate these functions and moreover, that insistence upon satisfaction of the matching rule can in some cases be used to greatly improve the rapidity of convergence of series involving logarithmic functions. The report emphasizes the very widespread applicability of the MAE method to problems in classical and modern, linear and nonlinear acoustics and related fields.

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Many problems in acoustics contain a small dimensionless parameter ϵ, and it is useful, both conceptually and from the point of view of numerical computation, to seek a solution in the form of a perturbation series. In the simplest cases the series would proceed by integral powers of ϵ,

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1. INTRODUCTION

Singular perturbation techniques have, over the past 25 years, been extensively applied to problems in fluid dynamics, the field in which many of these techniques were first introduced and in which most of their development has taken place. The literature relating to the method of Matched Asymptotic Expansions (MAE) alone is vast, running to six books [1-6], several dozen review articles and more than three thousand research papers, as at mid 1976. Despite the prolixity of these papers in many branches of steady and unsteady fluid mechanics, it is indeed surprising that modern perturbation methods should have made so little impact on one particular branch of unsteady fluid mechanics - Acoustics. It is even more surprising when one recalls the work of the great 19th Century workers in acoustics - Helmholtz, Kelvin, Stokes, Rayleigh - much of which has many ideas in common with those of the modern techniques, a point to which we shall return later.

There seem to be two principal reasons why acoustics has suffered in this respect, despite having been, a hundred years ago, the field in which MAE almost started. These are firstly, that classical acoustics appeared by around 1910 to be a fairly well worked out subject, largely because of the great work done in the second half of the 19th Century by Rayleigh and others. There were, of course, difficult diffraction problems (such as the quarter-plane diffraction problem) for which the mathematical techniques of 1910 were quite inadequate and for which even today they are hardly adequate. But the physical principles of classical acoustics seemed well enough understood, and as a result research in acoustics was taken by physicists into directions ever more remote from the studies of the compressive wave behavior, at reasonable frequencies, of air and water, which had been the previous concern of acoustics. Now of course classical acoustics was not worked out by 1910, but it took another forty years until Lighthill's work on aerodynamic noise in the early 1950's for it to be realized that the most essential aspect of ordinary acoustics (the aspect of energy transfer from nonpropagating modes of one kind or another into propagating acoustic energy) was in fact missing.

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[1] Technical References are listed at the end of the text.

That aspect now pervades most work on ordinary acoustics, either explicitly or implicitly, and has been responsible for much of the great development which has taken place in acoustics in the last 25 years. Nonetheless, much of acoustics is still in the hands of physicists working in very different areas, and physicists have been conspicuously slow to make any use of the perturbation techniques which have been devised by engineers and applied mathematicians working in fluid dynamics.

A second group of workers has also been slow to use these techniques. Those are the workers in wave diffraction theory, a field in which mathematical rigor is often necessary and has in any case become customary. Against the usual background in diffraction theory (of proofs of the convergence of iterative solutions to integral equations, of the solution of problems with the aid of function-analytic arguments in one or more complex variable planes, etc.), techniques like MAE appear very suspect indeed. There is not much that can be proven at intermediate stages of the calculations and the validity of the final results can usually only be proven in rather simple cases. If one is not worried unduly by lack of mathematical rigor, however, the gains from using modern perturbation methods can be immense. I quote three examples, in the first of which at three successive annual Theoretical Mechanics Colloquia in the U.K. were presented (i) a rigorous derivation of two terms for the scattering cross-section in a high-frequency water wave problem, running to 50 pages of print; (ii) an equally rigorous derivation of the same two terms by a more efficient method requiring 24 pages of print; (iii) use of MAE to find, in 2 pages of print, four terms of the expansion for the cross-section and to detect an error of sign in the second term of (i) and (ii). As the second example, Dr. Frank Leppington and I have used MAE, an approximate method, to derive *exact* closed form solutions for certain quantities in a thick plate diffraction problem [7] which previously had only been given approximately using a so-called exact modification of the Wiener-Hopf technique. As the third, asymptotic expansions for certain modified Mathieu functions were needed [8] in connection with a study of the behavior of a fluid jet of elliptic cross-section – modelling the kind of exhaust jet which propels the CONCORDE SST. I used MAE to find these expansions, which turned out to be new, and which have since been established rigorously by one of my colleagues, Dr. W. Barrett, of Leeds University, England.

I hope in these notes (a) to convince the reader that singular perturbation techniques, and in particular MAE, form a particularly valuable tool in the context of acoustic problems and (b) to show through simple examples in acoustics how MAE can be applied in a reasonably systematic and safe way. Dr. Martin Lesser and I have attempted this elsewhere [9] in an article specifically on MAE in acoustics. In retrospect, however, that article seems to involve rather too much formalism in its attempt to make the method routine, and seems also to deal with too wide a range of problems for the kind of course needed here. Accordingly, these notes will be restricted to deal with a few examples in detail, after which reference to the paper [9] may be valuable. Our main aim will be to establish firmly the basic ideas of the MAE technique in the simplest relevant context.

2. HEURISTIC APPROACH TO MATCHING AND ITS PITFALLS

A heuristic approach to matching has been widely used in the past. Our purpose here is to show how simple acoustic problems demonstrate that *anything short of a proper mathematical rule for matching is likely to lead to erroneous results*, even if the physical basis for heuristic matching is clear. The correct solutions will be found later with the aid of an Asymptotic Matching Principle.

We start by looking at the solution given by Landau and Lifschitz [10] for the sound field radiated into still fluid by the forced oscillation of a body about a fixed mean position. If the velocity of the (rigid) body is $\underline{V}\exp(-i\omega t)$, we want a solution of

$$\left.\begin{aligned} &(\nabla^2 + k_0^2)\phi = 0 \\ \textit{with} \quad &\phi \sim r^{-1}\exp(ik_0 r)\, f(\theta, \psi) \text{ as } r \to \infty \\ \text{and} \quad &\underline{n}\cdot\nabla\phi = \underline{n}\cdot\underline{V} \text{ on the body} \end{aligned}\right\} \qquad (2.1)$$

$k_0 = \omega/c_0$ being the acoustic wavenumber at frequency ω. If the body, of typical dimension L say, is *compact* - of small extent compared with the wavelength $2\pi k_0^{-1}$ - then we argue that, with axes fixed on an origin somewhere in the body, the motion is incompressible within a wavelength of the body. Thus for $r \ll k_0^{-1}$ we have

$$\left.\begin{aligned} &\nabla^2\phi = 0 \\ \text{and} \quad &\underline{n}\cdot\nabla\phi = \underline{n}\cdot\underline{V}, \end{aligned}\right\} \qquad (2.2)$$

the radiation condition being inapplicable for this range of r. It is argued, however, that ϕ cannot grow when $r \gg L$ and therefore that no eigensolution, which would necessarily grow algebraically for $r \gg L$, can be present. (The term eigensolution is used for any solution of $\nabla^2\phi = 0$ with $\underline{n}\cdot\nabla\phi = 0$ on the body. For a sphere of radius L, for example, the general axisymmetric eigensolution is

$$\phi = \sum_{n=1}^{\infty} A_n P_n(\cos\theta)\left\{r^n + \frac{n\,L^{2n+1}}{(n+1)\,r^{n+1}}\right\}$$

where P_n denotes the nth Legendre polynomial.) Then for $r \gg L$ this inner potential ϕ has an harmonic multipole representation, in which the monopole must be absent

$$\phi \sim a_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) + a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) + \ldots. , \tag{2.3}$$

the leading term of which is a potential dipole,

$$\phi \sim a_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) . \tag{2.4}$$

For example, in the case of a sphere of radius L the boundary condition on ϕ is

$$\frac{\partial \phi}{\partial r} = V \cos\theta \quad \text{on } r = L$$

if the velocity $\underline{V}$ is in direction $\theta = 0$, and the potential is

$$\phi = -\frac{VL^3}{2r^2} \cos\theta$$

which is precisely (2.3) if $\underline{a} = (1/2\ VL^3, 0, 0)$ and $a_{ij} = \ldots. = 0$.

Now we argue in a complementary fashion that *if* $r \gg L$ *then the body appears, to* leading order, as a singularity at the origin, so that for $r \gg L$ we seek a solution of the full Helmholtz equation, satisfying the radiation condition, with singular behavior at $r = 0$. The most general such solution is

$$\phi = b \frac{e^{ik_o r}}{r} + b_i \frac{\partial}{\partial x_i}\left(\frac{e^{ik_o r}}{r}\right) + b_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{e^{ik_o r}}{r}\right) + \ldots \tag{2.5}$$

$$\sim \frac{b}{r} + b_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) + \ldots$$

when $r \ll k_o^{-1}$. Thus in an *overlap domain*

$$L \ll r \ll k_o^{-1}$$

appropriate terms from

$$\frac{b}{r} + b_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) + \ldots$$

and $$a_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) + \ldots$$

should be identical. The obvious choice is

$$b = 0,\ a_i = b_i, \tag{2.6}$$

and thus, since $\underline{a}$ is known in terms of $\underline{V}$ for a body of given shape, we arrive at the leading order wave field in the form of an *acoustic dipole*

$$\phi \sim a_i \frac{\partial}{\partial x_i}\left(\frac{e^{ik_o r}}{r}\right)$$

$$\sim ik_o \frac{(\underline{a} \cdot \underline{x})}{r^2} \exp(ik_o r) \tag{2.7}$$

This result is in fact correct for a compact body of any shape. The argument is presented with such authority by Landau and Lifschitz [10] that one is only inclined to examine its weak points when one uses the same argument only to arrive at a plainly wrong result. Although this simple kind of argument was used a lot by Lord Rayleigh [11] and by Lamb [12] (to examine sound transmission through rows of parallel slits in a screen, for example), it is not difficult to find simple problems in which a strictly comparable result goes badly wrong, and we now give two such examples.

Take, for instance, the case of plane wave scattering by the said compact body. If the incident potential is

$$\phi^i = \exp ik_o x$$

then the scattered field ϕ satisfies

$$(\nabla^2 + k_o^2)\,\phi = 0$$

$$\phi \sim r^{-1} \exp(ik_o r)\, f(\theta) \text{ as } r \to \infty \tag{2.8}$$

$$\underline{n} \cdot \nabla\phi = -\underline{n} \cdot \nabla\phi^i \text{ on the body}$$

which is almost the problem (2.1) again. For $r \ll k_o^{-1}$ the potential is harmonic and the incident stream is effectively a uniform stream, $\phi^i \sim (ik_o)x$ and therefore in the case of a sphere the leading order inner field is precisely as before if we make the substitution $V = -ik_o$. Whether or not the scatterer is a sphere, the inner potential still has the form (2.3), dominated for $r \gg L$ by an harmonic dipole. Again, we can evidently "match" this dipole to an outer wave-field acoustic dipole

$$\phi \sim b_i \frac{\partial}{\partial x_i}\left(\frac{e^{ik_o r}}{r}\right)$$

by choosing the strength b_i of the acoustic dipole to agree with the a_i of the harmonic dipole in (2.4).

This time, however, the casual approach is quite inadequate. In fact the directivity function for the leading order wave field scattered from a compact sphere is

$$f(\theta) = 2 - 3 \cos\theta$$

i.e., the sum of equally important dipole *and monopole* terms. The correct monopole strength cannot be found from first-order matching alone, for the dipole is more singular than the monopole as $r \rightarrow 0$, and so dominates at leading order as we come in towards the scatterer. The monopole strength is in fact determined by *compressibility* effects in the small scattering region, expressed analytically by the next term in the expression near the body. We shall comment on this again after looking at this scattering problem in detail with MAE.

The Landau-Lifschitz problem of sound emission by an oscillating body also goes wrong when there is a uniform mean flow past the body. Then the answer one gets depends on whether pressures, or potentials - or some other field variables - are matched. For again, the outer behavior of the potential perturbation near the body is a dipole

$$\phi \sim a_i(t) \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) \tag{2.9}$$

and the leading term for the pressure ($p = -\rho(\partial/\partial t + U\,\partial/\partial x)\,\phi$) is

$$p \sim -\rho\, \dot{a}_i(t) \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) \tag{2.10}$$

It is argued that this should be matched to the inner limit of a solution to the *convected wave equation*, for when $r \gg L$ the steady perturbation due to the body is small and only the uniform flow at speed U remains. For simplicity suppose that the free stream Mach number $M = U/c_o$ is small, so that we can neglect M^2 compared with unity. Then a wave-field ϕ which agrees with (2.9) in $L \ll r \ll k_o^{-1}$ is

$$\phi = \frac{\partial}{\partial x_i}\left(\frac{a_i\left(t - \frac{r - Mx}{c_o}\right)}{r}\right) \tag{2.11}$$

The associated pressure in the wave field is then

$$p = -\rho \left(\frac{\partial}{\partial t} + Mc_o \frac{\partial}{\partial x}\right) \frac{\partial}{\partial x_i} \left(\frac{a_i \left(t - \frac{r - Mx}{c_o}\right)}{r} \right)$$

and for $r \ll k_o^{-1}$ that has the inner limit

$$p \sim -\rho\, \dot{a}_i(t) \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right)(1 - M\cos\theta) \qquad (2.12)$$

where $x = r\cos\theta$ and the stream is in the positive x-direction. Equation (2.12) does not match (2.10), but differs from it by a Doppler factor $(1 - M\cos\theta)$. Similar differences arise if one matches the pressures and then considers whether the velocities agree.

The message of this section is then simply this: that even to leading order some care is needed in ordering terms correctly, and a mathematical rule for matching must be devised and observed without the use of physical reasoning to shortcut any steps. I have heard it said that singular perturbation techniques were not in fact invented where and when popularly supposed (e.g., in Cal Tech in the 40's and 50's) but by such men as Rayleigh nearly a hundred years ago. I think the examples given here show that that is not a tenable view. The great 19th Century physicists discovered a physically appealing idea with something in common with modern methods; but they did not discover the mathematical formalism for turning their ideas into a reliable method, and perhaps did not appreciate the possibility of errors arising in the way I have tried to indicate. It is regrettable that the great book by Landau and Lifschitz should advocate the physical rather than the mathematical approach, but it is interesting to note that they attack the scattering problem by a completely different method (which of course gives the right answer).

3. FORMAL APPROACH TO MATCHING

It is a fairly general criterion that a perturbation problem (a problem containing a parameter ϵ where we are interested in the approximate behavior of the solution as ϵ tends to some value, usually zero or infinity) is *singular* and singular in a way calling for the use of MAE, if the problem involves two dynamically significant length scales whose ratio becomes either large or small as ϵ approaches its limiting value.

Low-frequency acoustic generation and scattering problems exemplify perfectly this kind of singular perturbation problem and its treatment by MAE. (In one problem [7], to which we shall return later, an acoustic problem served to highlight an extremely subtle, though probably common, failure of the Asymptotic Matching Rule as widely practiced in the literature and to show how a modified rule could be devised to rectify matters.) The perturbation parameter is the Helmholtz number $\epsilon = k_0 L$ which tends to zero, and the two obviously significant length scales, L characterizing the body geometry and k_0^{-1} characterizing the wave propagation, become asymptotically disparate as $\epsilon \to 0$.

Consider for definiteness a scattering problem, in which r' denotes the dimensional position variable and $\phi'(r', k_0, L)$ is the scattered potential for an incident potential $\exp(ik_0 x')$. Throughout this section the additional dependence on angular variables (θ, ψ) will be suppressed. To assess the smallness of various terms as $\epsilon \to 0$ we need dimensionless variables. The potential is already dimensionless, and either L or k_0^{-1} can be used to normalize lengths. We shall use the following notation

$r = k_0 r'$ denotes the *outer* position coordinate

$\phi'\left(r' = \frac{r}{k_0}, k_0, L\right) \equiv \phi(r, \epsilon)$ denotes the outer potential

$R = \frac{r'}{L} = \frac{r}{\epsilon}$ denotes the *inner* position coordinate

$\phi'(r' = LR, k_0, L) \equiv \Phi(R, \epsilon)$ denotes the inner potential

We want to know how $\phi(r, \epsilon)$ behaves as $\epsilon \to 0$ for *all* values of r from ϵ to ∞, or equivalently how $\Phi(R, \epsilon)$ behaves as $\epsilon \to 0$ for all R from 1 to ∞, because in general one needs to know the directivity of the scattered field at infinity and also the pressure on the scattering body. It is in the nature of a singular perturbation problem that it is necessarily *impossible* to find a single Poincaré asymptotic expansion, which in the simplest case would be a power series in ϵ,

$$\phi(r, \epsilon) \sim \phi_0(r) + \epsilon\phi_1(r) + \epsilon^2 \phi_2(r) + \ldots \tag{3.1}$$

which holds over the whole range of values of r of interest. This is because when such a series is inserted into differential equation and boundary conditions and like powers of ϵ are equated one has to assume that r is fixed and $O(1)$. Usually in these problems r can be allowed to increase indefinitely without affecting the process of equating powers of ϵ, and so boundary conditions at infinity - the radiation condition, for example - can be imposed on the functions $\phi_0, \phi_1, \ldots$ *If r is allowed to become small, however, the whole basis for the* expansion (3.1) is undermined as the terms then have different orders of magnitude from those postulated in the expansion. In particular this means that the terms $\phi_0, \phi_1, \ldots$ cannot usually be made to satisfy the required boundary condition on the body, where $r = O(\epsilon)$.

Consequently:

(a) We cannot find the pressure on the scattering body from the outer expansion

$$\phi(r, \epsilon) \sim \phi_0(r) + \epsilon\phi_1(r) + \epsilon^2 \phi_2(r) + \ldots$$

which is unlikely to hold when r is as small as ϵ.

(b) We cannot at the moment find even the far-field directivity from the outer expansion, because that expansion contains undetermined constants, and even functions, arising from the fact that only one boundary condition (at infinity) can be applied to the outer series.

Similar considerations apply to the inner potential which also, in the simplest case, might have a power series expansion

$$\Phi(R, \epsilon) \sim \Phi_0(R) + \epsilon\Phi_1(R) + \epsilon^2 \Phi_2(R) + \ldots \tag{3.2}$$

Formal processes (substitution in equations, equating powers of ϵ, etc.) will lead to a partial determination of Φ_0, Φ_1, etc., only for values of R of order unity. One expects therefore to be able to apply the boundary condition on the body, but not the condition at infinity, so that we can certainly not find the directivity function from the inner expansion nor can we at the moment even find the pressure on the scatterer from that inner expansion because of the presence of undetermined constants and functions in $\Phi_0, \Phi_1, \ldots$

The inner and outer expansions are not *uniformly valid.* The outer holds in the wave-field $r = 0(1)$ and usually right out to infinity. It may also hold for some smaller values of r, down to $r = 0(\epsilon^{1/2})$ perhaps, for example, but it will not usually hold uniformly down to values as small as $r = 0(\epsilon)$. The inner holds around the body, $R = 0(1)$, and perhaps also for larger values of R, up to $R = 0(\epsilon^{-1/4})$ say, but is unlikely to hold for values as large as $0(\epsilon^{-1})$ which are in the wave zone where r is the natural variable.

Now the inner and outer series are just different approximate representations for the *same* function $\phi'(r', k_0, L)$, so the question is "Can that fact be utilized to pin down the indeterminacies which exist in the inner or outer expansions separately, and so to find approximate solutions covering the whole range of interest?"

An affirmative answer to that question can be given for the case when the inner and outer expansions *overlap strongly,* that is, when both are simultaneously valid for a range of intermediate values of r, $r = 0(\epsilon^{\alpha})$ say, where

$$0 \leqslant \alpha_1 < \alpha < \alpha_2 \leqslant 1 \tag{3.3}$$

The number α_2 is connected with the smallest value of r for which the outer expansion holds, while α_1 is connected with the largest value of r for which the inner expansion holds. In the example just quoted, the outer expansion held down to $r = 0(\epsilon^{1/2})$, so that $\alpha_2 = 1/2$, the inner up to $R = 0(\epsilon^{-1/4})$, this corresponding to $r = 0(\epsilon^{3/4})$ and giving $\alpha_1 = 3/4$. Since $\alpha_1 > \alpha_2$ there is no overlap in this case, there is no known way of "matching" the inner and outer expansions, and indeed no general method may exist at all. If the inner holds up to $R = 0(\epsilon^{-1/2})$, i.e., to $r = 0(\epsilon^{1/2})$, then $\alpha_1 = \alpha_2 = 1/2$ and there is only *marginal overlap* along the line $\alpha = 1/2$ rather than in a domain $\alpha_1 < \alpha < \alpha_2$. In this case it may or may not be possible to match the expansions. No general rule is known and examples can be given of both possibilities [7, 13]. If, on the other hand, the outer holds down to $r = 0(\epsilon^{5/8})$ say, the inner up to $R = 0(\epsilon^{-1/2})$ then there is strong overlap in the domain $r = 0(\epsilon^{\alpha})$ where

$$1/2 < \alpha < 5/8$$

Then there is a matching rule, which, in the simpler problems at any rate, is sufficient to uniquely determine the various unknowns in the inner and outer expansions.

One version of the rule runs as follows (the *Intermediate Matching Principle*). Suppose one knows in advance that an overlap domain $r = 0(\epsilon^{\alpha})$ exists for α given by (3.3). (We should note that in general the values α_1 and α_2 will depend upon the orders to which the inner and outer expansions, respectively, are carried; α_1 will be a non-decreasing function of the number of terms retained in the inner expansion, α_2 will be a non-increasing function of the number retained in the outer series, so that the width of the overlap domain will, if anything, reduce as the expansions are taken to higher order.) Then one introduces an intermediate variable r_* by $r = \epsilon^{\alpha} r_*$, $R = r_*/\epsilon^{1-\alpha}$, so that $r \to 0$ and $R \to \infty$ as $\epsilon \to 0$ with r_* fixed. Inner and outer series are then expanded as $\epsilon \to 0$ for fixed values of r_* and corresponding terms, functions of r_* and ϵ, are made to agree. If one does not know the range of permissible values of α in advance one has to proceed tentatively, introducing the intermediate variable and trying to see whether there is a range of values for α which will allow the intermediate expansions of the inner and outer series to be matched term by term.

This is, at present, the most fundamental of the available matching procedures, though probably not the most common nor the simplest to use in practice. Difficulties arise, for instance, when the inner and outer expansions are re-expanded in terms of the intermediate variable r_*, it not being clear how many terms in the expansion are to be retained unless information is to hand on the domain of validity of the inner and outer series. The book by Cole [2] gives an extensive illustration of the use of the intermediate matching idea, although it avoids the difficulties inherent in this approach when carried to high order.

The most common procedure, and in my experience by far the most straightforward to use in a routine fashion, is a version of the principle put forward by Van Dyke [1] as the *Asymptotic Matching Principle.* In its original version it is ambiguous in some cases, so we propose here, as in [7], to use the following notation and principles:

We write

$$\phi^{(m)}(r, \epsilon) = \phi_o(r) + \epsilon\phi_1(r) + \ldots. + \epsilon^m \phi_m(r) \tag{3.4}$$

for the outer expansion truncated beyond $0(\epsilon^m)$. If the outer series really is asymptotic, then

$$| \phi(r, \epsilon) - \phi^{(m)}(r, \epsilon) | = o(\epsilon^m) \tag{3.5}$$

for appropriate values of r. Now we write $r = \epsilon R$ in $\phi^{(m)}$, hold R fixed and expand through terms $0(\epsilon^n)$ say. This gives a perfectly definite and readily calculated set of terms which will be denoted by

$$\phi^{(m, n)} \tag{3.6}$$

Next we take the inner expansion and truncate it beyond $0(\epsilon^n)$, giving

$$\Phi^{(n)}(R, \epsilon) = \Phi_o(R) + \ldots + \epsilon^n \Phi_n(R). \tag{3.7}$$

In this we make the transformation $R = r/\epsilon$, keep r fixed and expand as $\epsilon \to 0$ through terms $0(\epsilon^m)$, giving another set of terms denoted by

$$\Phi^{(n, m)} \tag{3.8}$$

As things now stand $\phi^{(m, n)}$ is a function of R, ϵ, $\Phi^{(n, m)}$ a function of r, ϵ, but we shall understand that both are expressed *without any further expansion or approximation* in terms of, say, r, ϵ.

Then the Asymptotic Matching Principle states that

$$\phi^{(m, n)} \equiv \Phi^{(n, m)} \tag{3.9}$$

It is easy to believe the truth of the principle on the basis that each side of (3.9) is an asymptotic representation for the potential in the overlap domain, and asymptotic representation with respect to a given set of gauge functions like integral powers of ϵ is unique, so that the two sides of (3.9) must be identical. That reasoning is generally quite false, though the conclusion as expressed in (3.9) is correct. Although $\phi^{(m)}$ and $\Phi^{(n)}$ *do* have asymptotic significance with respect to ϕ', the functions $\phi^{(m, n)}$ and $\Phi^{(n, m)}$ in general have *no asymptotic significance* whatever. This point is perhaps made clearer if we consider unequal values of m and n, say $m > n$. Then it appears that genuine information has been thrown away in forming $\phi^{(m,n)}$ from $\phi^{(m)}$, while information of no significance has been retained in forming $\Phi^{(n,m)}$ from $\Phi^{(n)}$. The principle (3.9) must not, however, be regarded as saying anything about the common asymptotics of the two expansions; it merely defines two functions associated with the expansions in a clearly prescribed way, and asserts their equality.

As far as justification of the MAE procedures goes, one can say the following. It is necessary first to prove that the formal substitution of an assumed series like (3.1, 3.2) into equations and boundary conditions will produce an asymptotic series for any values of r, R. This is a difficulty common to all perturbation methods, and to many other processes (solution of differential equations by series and contour integrals, for example) as well, and is not a difficulty peculiar to MAE. Under this heading one needs also to establish the range of values of r, R for which the outer and inner series are asymptotically valid, and thus to find the overlap domain, if it exists. Proofs of these points can be given in some cases, but the proofs fall far short of covering the kinds of problems of immediate research interest. Granted the resolution of these difficulties, however, the two overlapping expansions *must* match according to (3.9). This has been proven [6, 7, 13] under various different assumptions about the overlap, and with greater generality than we have so far required. The proof given in [7] involves showing that there exist numbers (m_1, n_1) such that all the terms inside the (m, n) "block" (the one of interest) plus some other terms outside the (m, n) block but inside the (m_1, n_1) block together constitute an asymptotic representation for ϕ' with an error smaller than the smallest term in the (m, n) block and for some intermediate range of values of r. This is done for both ϕ and Φ and the two asymptotic representations are equated term by term. The rule (3.9) is just a statement of this term by term equality for a particularly efficiently calculated block of terms.

The Asymptotic Matching Principle has been proven also under the following more general conditions, which are sufficient to cover most applications. Suppose that the inner and outer expansions proceed in fractional or integral powers of ϵ multiplied by integral powers of $\ell n\epsilon$,

$$\phi(r,\epsilon) \sim \phi_0(r) + \epsilon \ell n\epsilon\ \phi_1(r) + \epsilon\phi_2(r) + \epsilon^2 \ell n^2\epsilon\ \phi_3(r) + \epsilon^2 \ell n\epsilon\ \phi_4(r) + \epsilon^2\phi_5(r) + \epsilon^3 \ell n^3\epsilon\ \phi_6(r) + \ldots \tag{3.10}$$

$$\Phi(R,\epsilon) \sim \epsilon^{1/2}\Phi_0(R) + \epsilon\Phi_1(R) + \epsilon^{3/2}\ell n\epsilon\ \Phi_2(R) + \epsilon^{3/2}\Phi_3(R) + \epsilon^2 \ell n\epsilon\ \Phi_4(R) + \epsilon^2\Phi_5(R) + \epsilon^{5/2}\ell n^2\epsilon\ \phi_6(R) + \ldots \tag{3.11}$$

being a case that arises in diffraction by a thick rigid plate [7] and is interesting in several respects. Then a function like $\phi^{(m)}$ or $\Phi^{(n)}$ will be constructed as before, by terminating the expansion beyond $O(\epsilon^m)$ or $O(\epsilon^n)$ as the case may be, where now m and n can be any rational numbers which may or may not appear in either of the series. In truncating the series, all logarithmic terms are to be grouped together according to the order of their algebraic multiplier, and are never to be separated out. Thus, with the above forms for example, we have

$$\phi^{(1)} = \phi_o + \epsilon \ln \epsilon \; \phi_1 + \epsilon \phi_2$$

$$\phi^{(1/2)} = \phi_o ,$$

$$\Phi^{(1)} = \epsilon^{1/2} \Phi_o + \epsilon \Phi_1$$

$$\Phi^{(3/2)} = \epsilon^{1/2} \Phi_o + \epsilon \Phi_1 + \epsilon^{3/2} \ln \epsilon \; \Phi_2 + \epsilon^{3/2} \Phi_3$$

Having taken any two rational numbers m and n, and formed $\phi^{(m)}$ and $\Phi^{(n)}$, one then writes $r = \epsilon R$ in $\phi^{(m)}$ and expands up to and including all terms which are not smaller than ϵ^n, writes $R = r/\epsilon$ in $\Phi^{(n)}$ and expands keeping all terms not smaller than ϵ^m, giving the quantities $\phi^{(m,n)}$ and $\Phi^{(n,m)}$. *Again, algebraic order only* is taken into account, $\epsilon^\lambda \ln^\mu \epsilon$ being regarded as $O(\epsilon^\lambda)$ whatever the value of μ. Then the matching principle

$$\phi^{(m,n)} \equiv \Phi^{(n,m)} \tag{3.12}$$

has been established also [7, 13] for expansions of this kind.

Occasionally it is necessary to generalize this rule further, as for example in the case of scattering by a soft body [9]. But that example deserves separate study for other reasons, and for most problems the principle (3.12) is adequate.

4. SOUND GENERATION BY FORCED OSCILLATIONS

We return now to the problem of sound generation by the forced oscillation of a rigid compact body. For definiteness we take a sphere of radius L whose velocity in the positive x′-direction ($\theta = 0$) is $V \exp(-i\omega t)$, so that we need an axisymmetric solution of

$$\left.\begin{aligned} (\nabla'^2 + k_o^2)\,\phi' &= 0 \\ \phi' &\sim r'^{-1} \exp(ik_o r')\, f(\theta) \quad \text{as } r' \to \infty \\ \frac{\partial \phi'}{\partial r'} &= V \cos\theta \quad \text{on } r' = L \end{aligned}\right\} \tag{4.1}$$

In terms of the outer variable $r = k_o r'$ and with an arbitrary normalization for ϕ', say $\phi' = VL\phi$, the complete problem is

$$\left.\begin{aligned}
&(\nabla_r^2 + 1)\,\phi = 0 \\
&\phi \sim r^{-1}\exp(ir)\,g(\theta) \quad \text{as } r \to \infty \\
&\frac{\partial\phi}{\partial r} = \frac{1}{\epsilon}\cos\theta \quad \text{on } r = \epsilon
\end{aligned}\right\} \tag{4.2}$$

with $\epsilon = k_o L$. Letting $\epsilon \to 0$ makes the body boundary condition tell us no more than that ϕ must be singular in some fashion as $r \to 0$. Whatever the form taken by the outer expansion (and that cannot usually be assumed in advance, for it is dictated to some extent by information from the inner region) all terms in the outer expansion must therefore be radiating axisymmetric solutions of the Helmholtz equation, the general solution of which is a multipole series,

$$a\left(\frac{e^{ir}}{r}\right) + a_i \frac{\partial}{\partial x_i}\left(\frac{e^{ir}}{r}\right) + a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{e^{ir}}{r}\right) + \ldots. \tag{4.3}$$

A little thought shows that the most general *axisymmetric* solution must be

$$a\left(\frac{e^{ir}}{r}\right) + a_1\frac{\partial}{\partial x}\left(\frac{e^{ir}}{r}\right) + a_{11}\frac{\partial^2}{\partial x^2}\left(\frac{e^{ir}}{r}\right) + a_{22}\nabla_\perp^2\left(\frac{e^{ir}}{r}\right) + a_{111}\frac{\partial^3}{\partial x^3}\left(\frac{e^{ir}}{r}\right)$$

$$+ a_{122}\frac{\partial}{\partial x}\nabla_\perp^2\left(\frac{e^{ir}}{r}\right) + \text{(higher multipoles than octupoles)} \tag{4.4}$$

Let us assume therefore that the outer series starts off with a term of algebraic order, $\epsilon^\lambda\phi_o$ say, where ϕ_o has the above form, in which $\nabla_\perp^2$ denotes the Laplace operator in the two transverse coordinates,

$$\nabla_\perp^2 \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The coefficients a, a_1, a_{11}, etc. will then be given a superscript (o) to indicate that they refer to ϕ_o.

We can also simplify (4.4) somewhat, using the fact that for $r \neq 0$

$$\left(\nabla_\perp^2 + \frac{\partial^2}{\partial x^2} + 1\right)\left(\frac{e^{ir}}{r}\right) = 0$$

and therefore the terms with coefficients a_{22} and a_{122} can be assumed to be already included elsewhere.

Turn now to the inner problem, obtained from (4.2) by writing $\Phi(R, \epsilon)$ for $\phi(r, \epsilon)$ and $r = \epsilon R$, so that

$$\left.\begin{aligned} &(\nabla_R^2 + \epsilon^2)\,\Phi = 0 \\ &\Phi \sim R^{-1} \exp(i\epsilon R)\, h(\theta) \quad \text{as } \epsilon R \to \infty \\ &\frac{\partial \Phi}{\partial R} = \cos\theta \quad \text{on } R = 1 \end{aligned}\right\} \tag{4.5}$$

the radiation condition obviously being unenforceable in the limit $\epsilon \to 0$.

The body boundary condition indicates that (with the chosen normalization, $\phi' = VL\Phi$) Φ is of order unity in the inner region, so that the inner series should take the form

$$\Phi \sim \Phi_o + (\text{terms which vanish with } \epsilon).$$

(The only other possibility would be that Φ starts with a term which becomes infinite as $\epsilon \to 0$, $\Phi \sim \epsilon^{-1}\,\Phi_{-1}$ say. Then Φ_{-1} would have to be an *inner eigensolution*, the general axisymmetric form of which was given following equation (2.2). If one persists with this and attempts to match with the outer solution, one finds of course that $\Phi_{-1} \equiv 0$. The reader should verify this after seeing how the first nontrivial matching is carried out.) The problem for Φ_o is then

$$\left.\begin{aligned} &\nabla_R^2\, \Phi_o = 0 \\ &\frac{\partial \Phi_o}{\partial R} = \cos\theta \quad \text{on } R = 1 \end{aligned}\right\} \tag{4.6}$$

whose general solution is

$$\Phi_o = -\frac{1}{2R^2}\cos\theta + \sum_{\nu=1}^{\infty} A_\nu^{(o)}\, P_\nu(\cos\theta) \left\{ R^\nu + \frac{\nu}{\nu+1}\,\frac{1}{R^{\nu+1}} \right\} \tag{4.7}$$

the sum of a particular solution which decays at infinity, plus a general axisymmetric eigensolution satisfying

$$\nabla_R^2\, \Phi = 0, \ \frac{\partial \Phi}{\partial R} = 0 \quad \text{on } R = 1$$

For matching, note that when $\phi^{(\lambda)} \equiv \epsilon^{\lambda}\phi_o$ is expanded with $r = \epsilon R$, the leading term will involve only inverse powers of R, while when $\Phi^{(o)} \equiv \Phi_o$ is expanded with $R = r/\epsilon$, the leading term will involve the positive power r^{ν} unless we choose $A_{\nu}^{(o)} \equiv 0$. Thus no inner eigensolutions are needed to leading order (though they are needed at higher order to match the ascending powers of R which arise from the expansion of the phase factors $\exp(i\epsilon R)$ in (4.4)).

We then have

$$\Phi^{(o)} = \Phi_o = -\frac{1}{2R^2}\cos\theta$$

$$\Phi^{(o)}\left(\frac{r}{\epsilon}\right) = -\frac{\epsilon^2}{2r^2}\cos\theta$$

and so

$$\Phi^{(o,o)} = \Phi^{(o,1)} = 0$$

$$\Phi^{(o,2)} = -\frac{\epsilon^2}{2r^2}\cos\theta = \Phi^{(o,3)} = \Phi^{(o,4)} \ldots .$$

On the other hand, for the leading outer solution we have

$$\phi^{(\lambda)} = \epsilon^{\lambda}\phi_o$$

and $\phi^{(\lambda)}(\epsilon R) = 0(\epsilon^{\lambda - N})$ where N is the highest multipole order present in (4.4) (N = 1 monopole, N = 2 dipole, n = 3 quadrupole, N = 4 octupole, etc.). Suppose then that λ were equal to 0. Then $\phi^{(o)}(\epsilon R) = 0(\epsilon^{-1})$ at the very least, and because that term must match with $\Phi^{(o,o)} = 0$ the only possibility is that $\phi_o = 0$. The same conclusion is reached if we take $\lambda = 1$, because $\Phi^{(o,1)} = 0$. We therefore try taking $\lambda = 2$ – i.e., *the order of the leading outer term is that of the first nontrivial term in the outer expansion of the leading inner term* – so that we have

$$\phi^{(2)} = \epsilon^2\phi_o .$$

Since the expansion of $\Phi^{(o)}(r/\epsilon)$ contains no inverse power of ϵ it follows that only the values N = 1 and N = 2 are permitted in ϕ_o, and hence

$$\phi_o = a^{(o)}\left(\frac{e^{ir}}{r}\right) + a_1^{(o)}\frac{\partial}{\partial x}\left(\frac{e^{ir}}{r}\right), \qquad (4.8)$$

$$\left.\begin{aligned}
\phi^{(2)}(\epsilon R) &= \epsilon a^{(o)}\left(\frac{e^{i\epsilon R}}{R}\right) + a_1^{(o)}\frac{\partial}{\partial X}\left(\frac{e^{i\epsilon R}}{R}\right), \\
\phi^{(2,o)} &= a_1^{(o)}\frac{\partial}{\partial X}\left(\frac{1}{R}\right) = -a_1^{(o)}\frac{1}{R^2}\cos\theta \\
\phi^{(2,1)} &= \epsilon a^{(o)}\left(\frac{1}{R}\right) + \phi^{(2,o)} \\
\phi^{(2,2)} &= -\epsilon^2 a_1^{(o)}\frac{\partial}{\partial X}\left(\frac{R}{2}\right) + \phi^{(2,1)} = -\epsilon^2\frac{a_1^{(o)}}{2}\cos\theta + \phi^{(2,1)}
\end{aligned}\right\} \tag{4.9}$$

From the matching

$$\phi^{(2,o)} \equiv \Phi^{(o,2)}$$

we thus get

$$a_1^{(o)} = \frac{1}{2} \tag{4.10}$$

which determines the dipole strength, but not the monopole $a^{(o)}$. We see, however, that this could be found from $\phi^{(2,1)}$ in (4.9) by matching to $\Phi^{(1,2)}$, and therefore, prompted by the fact that the inner expansion of $\phi^{(2)}(\epsilon R)$ contains terms $O(1)$ and $O(\epsilon)$, we try to continue the inner expansion with

$$\Phi \sim \Phi_o + \epsilon\Phi_1 + o(\epsilon). \tag{4.11}$$

Φ_1 is an inner eigensolution

$$\Phi_1 = \sum_{\nu=1}^{\infty} A_\nu^{(1)} P_\nu(\cos\theta)\left\{R^\nu + \left(\frac{\nu}{\nu+1}\right)\frac{1}{R^{\nu+1}}\right\} \tag{4.12}$$

and the expansion of

$$\Phi^{(1)}\left(\frac{r}{\epsilon}\right) = \Phi_o\left(\frac{r}{\epsilon}\right) + \epsilon\Phi_1\left(\frac{r}{\epsilon}\right)$$

must start with terms $O(1)$ and $O(\epsilon)$ in order that $\Phi^{(1,2)}$ can be matched to $\phi^{(2,1)}$ in (4.9). It follows that only the term with $\nu = 1$ can be present in (4.12) and that then

$$\Phi^{(1,2)} = -\frac{\epsilon^2}{2r^2}\cos\theta + A_1^{(1)} r\cos\theta \tag{4.13}$$

This is identical with

$$\phi^{(2,1)} = \epsilon a^{(o)} \frac{1}{R} - \frac{1}{2R^2} \cos\theta$$

if and only if

$$A_1^{(1)} = a^{(o)} = 0 \tag{4.14}$$

Thus matching of the second order inner solution to the first order outer solution shows that the second order inner solution is identically zero in this case, and that the previously undetermined leading order outer monopole strength is also zero. We have recovered the Landau-Lifschitz result (2.7), and the "obvious" choice (2.6) is seen to be correct - but for a far deeper reason.

At this stage we will break off and make a similar examination of the plane wave scattering problem, where we shall see that the second order inner problem is not trivial, as it was above, and this has a profound influence on the leading order outer field, making the so-called "obvious" choice (2.6) of zero monopole strength not only not obvious, but actually incorrect.

5. PLANE WAVE SCATTERING

For the scattering by a fixed rigid sphere of radius L of a plane incident wave of potential $\exp(ik_o x')$, the problem for the scattered field ϕ' is

$$\left.\begin{aligned} &(\nabla'^2 + k_o^2)\,\phi' = 0 \\ &\phi' \sim r'^{-1} \exp(ik_o r')\, f(\theta) \quad \text{as } r' \to \infty \\ &\frac{\partial \phi'}{\partial r'} = -ik_o \cos\theta \exp(i\epsilon\cos\theta) \quad \text{on } r' = L \end{aligned}\right\} \tag{5.1}$$

The potentials are already dimensionless and hence the problem in outer variables takes the form

$$\left.\begin{aligned} &(\nabla_r^2 + 1)\,\phi = 0 \\ &\phi \sim r^{-1} \exp(ir)\, g(\theta) \quad \text{as } r \to \infty \\ &\frac{\partial \phi}{\partial r} = -i \cos\theta \exp(i\epsilon\cos\theta) \quad \text{on } r = \epsilon \end{aligned}\right\} \tag{5.2}$$

while in inner variables we have

$$\left.\begin{aligned} &(\nabla_R^2 + \epsilon^2)\,\Phi = 0 \\ &\Phi \sim R^{-1}\,\exp(i\epsilon R)\,h(\theta) \quad \text{as } \epsilon R \to \infty \\ &\frac{\partial \Phi}{\partial R} = -i\epsilon\cos\theta\,\exp(i\epsilon\cos\theta) \quad \text{on } R = 1 \end{aligned}\right\} \tag{5.3}$$

As in §4, all outer solutions, to any order in ϵ, are outer ***eigensolutions***, with the general form (4.4), but the order λ of the leading outer potential, $\phi \sim \epsilon^{\lambda}\phi_0$, is not yet known. It can be argued, as before, that the order of the leading inner potential is that ***distinguished order*** (a term used by Cole [2]) for which the solution is not just an inner eigensolution. Thus

$$\Phi \sim \epsilon\Phi_0 + \ldots$$

where

$$\Phi_0 = +\frac{1}{2R^2}\cos\theta + \sum_{\nu=1}^{\infty} A_\nu^{(o)} P_\nu(\cos\theta)\left\{R^\nu + \left(\frac{\nu}{\nu+1}\right)\frac{1}{R^{\nu+1}}\right\} \tag{5.4}$$

and again, as in §4, all the $A_\nu^{(o)} \equiv 0$. We then find that λ has the value 3 this time, with monopoles and dipoles permitted in ϕ_0, so that

$$\phi \sim \epsilon^3\phi_0 + \ldots.$$

$$\phi_0 = a^{(o)}\left(\frac{e^{ir}}{r}\right) + a_1^{(o)}\frac{\partial}{\partial x}\left(\frac{e^{ir}}{r}\right) \tag{5.5}$$

It might be thought that, since λ is 3 here rather than 2, quadrupoles should also be permitted in ϕ_0 this time. That cannot be, however, for then $\epsilon^3\phi_0(\epsilon R)$ would contain an $O(1)$ term from the quadrupole, which could not be matched to the inner solution which is $O(\epsilon)$.

The matching rule

$$\phi^{(3,1)} \equiv \Phi^{(1,3)}$$

determines the dipole strength as

$$a_1^{(o)} = -\frac{i}{2} \tag{5.6}$$

but *again* fails to determine the monopole strength. We therefore move on to the second inner problem, the inner expansion continuing with

$$\Phi \sim \epsilon\Phi_o + \epsilon^2\Phi_1 + \ldots$$

The difference between the generation and scattering problems is apparent now, in that Φ_1 is not an eigensolution but a solution of

$$\left.\begin{aligned} \nabla_R^2\,\Phi_1 &= 0 \\ \frac{\partial\Phi_1}{\partial R} &= \cos^2\theta \quad \text{on } R = 1 \end{aligned}\right\} \qquad (5.7)$$

Writing the boundary condition as

$$\frac{\partial\Phi_1}{\partial R} = \frac{2}{3}\,P_2(\cos\theta) + \frac{1}{3}$$

we easily find a particular solution in the form

$$\Phi_1 = -\frac{2}{9}\,\frac{P_2(\cos\theta)}{R^3} - \frac{1}{3R}$$

to which is to be added the general eigen solution

$$\sum_{\nu=1}^{\infty} A_\nu^{(1)}\,P_\nu(\cos\theta)\left\{R^\nu + \left(\frac{\nu}{\nu+1}\right)\frac{1}{R^{\nu+1}}\right\}$$

Application of the matching rule

$$\phi^{(3,2)} \equiv \Phi^{(2,3)}$$

now determines the monopole strength as

$$a^{(o)} = -\frac{1}{3} \qquad (5.9)$$

and shows that $A_\nu^{(1)} \equiv 0$, so that the inner eigensolutions play no part in the second term also.

The leading order wave-field is then

$$\phi \sim \epsilon^3\left\{-\frac{1}{3}\,\frac{e^{ir}}{r} - \frac{i}{2}\,\frac{\partial}{\partial x}\left(\frac{e^{ir}}{r}\right)\right\}$$

from which the directivity function, as $r \to \infty$, then follows at once as proportional to 2-3 $\cos\theta$, as quoted in §2. Note that the monopole term in the far-field is directly comparable with the dipole term, yet it arises from a weak monopole $-\epsilon^2/3R$ in the near-field as compared with the dipole $i\epsilon \cos\theta/2R^2$. Such subtleties are the essence of Acoustics, even in the low frequency limit, and Acoustics is therefore an excellent vehicle for exposing aspects of techniques which are easily overlooked in apparently more complicated but basically less delicate fields.

The above procedure may seem rather laborious but we have gone into it in some detail because a routine framework of this kind is indispensible in more difficult problems. Moreover, with practice the way in which the expansions are likely to proceed can often be anticipated (though as Van Dyke [1] emphasizes, many workers have obtained incorrect results by failing to suspect the presence of certain kinds of terms, usually logarithmic terms, whereas adherence to the procedure used here will greatly reduce that risk) and it not always necessary to include the most general eigensolution if one keeps in mind the forms of the terms which need to be matched (though again, failure to find the most general eigensolution, particularly in nonlinear problems, has led to many incorrect results in the past, and *general* eigensolutions should be included if there is the slightest suspicion of difficulty).

6. HIGHER APPROXIMATIONS

We continue here with the scattering problem to obtain one more term of each series. The inner series so far is

$$\Phi \sim \epsilon\left\{\frac{i}{2R^2}\cos\theta\right\} - \epsilon^2\left\{\frac{2}{9}\,\frac{P_2(\cos\theta)}{R^3} + \frac{1}{3R}\right\} + o(\epsilon^2) \tag{6.1}$$

If $\Phi^{(2)}(r/\epsilon)$ is expanded for $\epsilon \to 0$, the first terms are $O(\epsilon^3)$ and $O(\epsilon^5)$, so we attempt to continue the outer expansion in the form

$$\phi \sim \epsilon^3\phi_0 + \epsilon^5\phi_1 + o(\epsilon^5) \tag{6.2}$$

using the matching rule

$$\Phi^{(2,5)} \equiv \phi^{(5,2)} \tag{6.3}$$

Since $\Phi = O(\epsilon)$, the expansion of $\phi(\epsilon R)$ cannot contain any terms larger than $O(\epsilon)$, and so ϕ_1 can contain monopole, dipole, quadrupole and octupole elements, with the general form

$$\phi_1 = a^{(1)}\left(\frac{e^{ir}}{r}\right) + a_1^{(1)}\frac{\partial}{\partial x}\left(\frac{e^{ir}}{r}\right) + a_{11}^{(1)}\frac{\partial^2}{\partial x^2}\left(\frac{e^{ir}}{r}\right) + a_{111}^{(1)}\frac{\partial^3}{\partial x^3}\left(\frac{e^{ir}}{r}\right) \tag{6.4}$$

Then

$$\phi^{(5)}\,(\epsilon R) = \epsilon^3 \phi_o\,(\epsilon R) + \epsilon^5 \phi_1\,(\epsilon R)$$

and expanding gives

$$\phi^{(5)}(\epsilon R) = \epsilon^3 \left\{ -\frac{1}{3}\,\frac{(1 + i\epsilon R - \frac{\epsilon^2 R^2}{2} + ..)}{\epsilon R} - \frac{i}{2\epsilon^2}\,\frac{\partial}{\partial X}\,\frac{(1 + i\epsilon R - \frac{\epsilon^2 R^2}{2} + \ldots)}{R} \right\}$$

$$+ \epsilon^5 \left\{ a^{(1)}\,\frac{(1 + i\epsilon R + \ldots)}{\epsilon R} + \frac{a_1^{(1)}}{\epsilon^2}\,\frac{\partial}{\partial X}\,\frac{(1 + i\epsilon R + \ldots)}{R} \right.$$

$$\left. + \frac{a_{11}^{(1)}}{\epsilon^3}\,\frac{\partial^2}{\partial X^2}\,\frac{(1 + i\epsilon R + ..)}{R} + \frac{a_{111}^{(1)}}{\epsilon^4}\,\frac{\partial^3}{\partial X^3}\,\frac{(1 + i\epsilon R + \ldots)}{R} \right\} \tag{6.5}$$

Truncating this expansion beyond $0(\epsilon^2)$ gives

$$\phi^{(5,2)} = -\frac{1}{3}\epsilon^2\,\frac{1}{R} - \frac{i\epsilon}{2}\,\frac{\partial}{\partial X}\left(\frac{1}{R}\right) + \epsilon^2\, a_{11}^{(1)}\,\frac{\partial^2}{\partial X^2}\left(\frac{1}{R}\right) + \epsilon\, a_{111}^{(1)}\,\frac{\partial^3}{\partial X^3}\left(\frac{1}{R}\right) \tag{6.6}$$

while from (6.1)

$$\Phi^{(2,5)} = \frac{i\epsilon^3}{2r^2}\cos\theta - \frac{2}{9}\epsilon^5\,\frac{P_2(\cos\theta)}{r^3} - \frac{1}{3}\epsilon^3\,\frac{1}{r}. \tag{6.7}$$

Express (6.6) and (6.7) in terms of the same variable and use the fact that $P_2(\cos\theta) = (1/2)(3\cos^2\theta - 1)$. Then the matching rule (6.3) requires

$$\frac{i\epsilon}{2R^2}\cos\theta - \frac{1}{9}\epsilon^2\,\frac{(3\cos^2\theta - 1)}{R^3} - \frac{1}{3}\epsilon^2\,\frac{1}{R} \equiv -\frac{1}{3}\epsilon^2\,\frac{1}{R} + \frac{i\epsilon}{2R^2}\cos\theta$$

$$+ \epsilon^2\, a_{11}^{(1)}\,\frac{(3\cos^2\theta - 1)}{R^3} + \epsilon\, a_{111}^{(1)}\left[\frac{9\cos\theta}{R^4} - \frac{15\cos^3\theta}{R^4}\right]$$

where the differentiations in (6.6) have been performed using $\partial R/\partial X = X/R = \cos\theta$. Thus matching gives

$$a_{11}^{(1)} = -\frac{1}{9}$$

$$a_{111}^{(1)} = 0, \tag{6.8}$$

and determines the most singular (octupole and quadrupole) terms in ϕ_1, but fails to determine the dipole and monopole terms.

To obtain these it is necessary to go further with the inner expansion. Consideration of (6.5) shows that the dipole term in (6.4), with coefficient $a_1^{(1)}$, makes a contribution to $\phi^{(5)}(\epsilon R)$ of order $0(\epsilon^3)$, while the monopole, with coefficient $a^{(1)}$ makes a contribution of $0(\epsilon^4)$. Therefore $a_1^{(1)}$ will be determined if we use the matching rule

$$\phi^{(5,3)} = \Phi^{(3,5)} \tag{6.9}$$

which needs the third term of the inner expansion, while the monopole can only be determined from the fourth term of the inner series and use of the rule

$$\phi^{(5,4)} = \Phi^{(4,5)} \tag{6.10}$$

Accordingly, we assume an inner expansion

$$\Phi \sim \epsilon\, \Phi_0 + \epsilon^2 \Phi_1 + \epsilon^3 \Phi_2 + \epsilon^4 \Phi_3 + \ldots \tag{6.11}$$

which gives the problem

$$\left.\begin{aligned} \nabla_R^2\, \Phi_2 &= -\Phi_0 = -\frac{i}{2R^2} \cos\theta \\ \frac{\partial \Phi_2}{\partial R} &= +\frac{i}{2} \cos^3\theta \quad \text{on } R = 1 \end{aligned}\right\} \tag{6.12}$$

for the third term. This is typical of the higher order inner problems; not only are the boundary conditions inhomogeneous (as they were for Φ_0 and Φ_1), but the Laplace equation itself now has a forcing function related to the lower order potentials. Here the form of the forcing suggests looking for a particular integral of the form $f(R) \cos\theta$, and then one quickly finds that $f(R) = (i/4)$. We therefore write

$$\Phi_2 = \frac{i}{4} \cos\theta + \psi \tag{6.13}$$

and then

$$\nabla_R^2\, \psi = 0$$

$$\begin{aligned} \frac{\partial \psi}{\partial R} &= +\frac{i}{2} \cos^3\theta \\ &= +\frac{i}{5} P_3(\cos\theta) + \frac{3i}{10} P_1(\cos\theta) \end{aligned} \tag{6.14}$$

where $P_1(\cos\theta) = \cos\theta$, $P_3(\cos\theta) = (1/2)(5\cos^3\theta - 3\cos\theta)$ and the transformation to Legendre functions is made so that we can most readily use the facts that $R^n P_n(\cos\theta)$ and $R^{-(n+1)} P_n(\cos\theta)$ are axisymmetric solutions of Laplace's equation (and are the only ones which are finite on the axis, where $\cos\theta = \pm 1$). We now try to eliminate the forcing terms from the boundary condition (6.14) by looking for a particular solution

$$\alpha \frac{P_3(\cos\theta)}{R^4} + \beta \frac{P_1(\cos\theta)}{R^2}$$

which gives $\alpha = -i/20$, $\beta = -3i/20$.

To this we have to add the general inner eigensolution, as given in (4.7) with coefficients $A_\nu^{(2)}$, so that the general solution for Φ_2 is

$$\Phi_2 = \frac{i}{4}\cos\theta - \frac{i}{20}\frac{P_3(\cos\theta)}{R^4} - \frac{3i}{20}\frac{P_1(\cos\theta)}{R^2}$$

$$+ \sum_{\nu=1}^{\infty} A_\nu^{(2)} P_\nu(\cos\theta)\left\{R^\nu + \left(\frac{\nu}{\nu+1}\right)\frac{1}{R^{\nu+1}}\right\}. \tag{6.15}$$

Note here that, whatever the values taken by the $A_\nu^{(2)}$, Φ_2 does *not* tend to zero as $R \to \infty$ and there is no way of making it do so. At still higher orders the inner potentials will not merely remain finite as $R \to \infty$, but will actually tend to infinity.

The function $\Phi^{(3)}$ *is now given by*

$$\Phi^{(3)}\left(\frac{r}{\epsilon}\right) = \frac{i\epsilon^3}{2r^2}\cos\theta - \frac{2}{9}\epsilon^5\frac{P_2(\cos\theta)}{r^3} - \frac{\epsilon^3}{3r} + \frac{i\epsilon^3}{4}\cos\theta - \frac{3i}{20}\epsilon^5\frac{P_1(\cos\theta)}{r^2}$$

$$- \frac{i}{20}\epsilon^7\frac{P_3(\cos\theta)}{r^4} + \epsilon^3\sum_{\nu=1}^{\infty} A_\nu^{(2)} P_\nu(\cos\theta)\left\{\left(\frac{r}{\epsilon}\right)^\nu + \left(\frac{\nu}{\nu+1}\right)\left(\frac{\epsilon}{r}\right)^{\nu+1}\right\} \tag{6.16}$$

in which the leading order terms must be at most $O(\epsilon^3)$ in order for this to be matched to the outer series which starts with $\epsilon^3\phi_0$. Therefore

$$A_\nu^{(2)} = 0 \tag{6.17}$$

and again it seems that the inner eigensolutions are "too singular" as $R \to \infty$ to be tolerated. We shall return to this in a moment. For matching according to (6.9) we have

$$\Phi^{(3,5)} = \epsilon \frac{i}{2R^2} \cos\theta - \frac{2}{9}\epsilon^2 \frac{P_2(\cos\theta)}{R^3} - \frac{\epsilon^2}{3R} + \frac{i\epsilon^3}{4}\cos\theta - \frac{3i}{20}\epsilon^3 \frac{P_1(\cos\theta)}{R^2}, \qquad (6.18)$$

after truncating (6.16) at $0(\epsilon^5)$ and *then* returning to the variable R, while from (6.5) we have

$$\phi^{(5,3)} = -\frac{1}{3}\frac{\epsilon^2}{R} - \frac{i\epsilon^3}{3} - \frac{i\epsilon}{2}\left[-\frac{\cos\theta}{R^2} - \frac{1}{2}\epsilon^2 \cos\theta\right]$$

$$-\epsilon^3 a_1^{(1)} \frac{1}{R^2}\cos\theta + \frac{1}{9}\epsilon^2\left[\frac{1}{R^3} - \frac{3\cos^2\theta}{R^3}\right] \qquad (6.19)$$

(use having been made of the results (6.8)).

The equality of (6.18) and (6.19) determines the dipole coefficient as

$$a_1^{(1)} = +\frac{3i}{20}, \qquad (6.20)$$

and all other terms in (6.19) have identical counterparts in (6.18) except for the term $-i\epsilon^3/3$. This outer term is unmatched because of a *failure to take the most general inner eigensolution* in (6.15). To that eigensolution should be added the *constant* term

$$A_o^{(2)}$$

which is all too easily discarded as irrelevant. It has a serious effect at higher order, however, because later the function Φ_2 will act as a forcing function for the inner term $\epsilon^5\Phi_4$, and the constant part $A_o^{(2)}$ of Φ_2 will induce a variable part of Φ_4 and hence, through the matchings, of the wave field too. It we add $A_o^{(2)}$ to (6.15), (6.18) acquires a term

$$\epsilon^3 A_o^{(2)}$$

which can be matched to the corresponding term in (6.19) provided

$$A_o^{(2)} = -\frac{i}{3}. \qquad (6.21)$$

Thus the lowest order (least rapidly growing) inner eigensolution is called into play in the third order inner solution.

To find Φ_3 in (6.11) we have the problem

$$\left.\begin{aligned} \nabla_R^2\, \Phi_3 &= -\Phi_1 = \frac{2}{9}\, \frac{P_2(\cos\theta)}{R^3} + \frac{1}{3R}, \\ \frac{\partial \Phi_3}{\partial R} &= -\frac{1}{6}\cos^4\theta \text{ on } R = 1 \end{aligned}\right\} \tag{6.22}$$

This is a complicated but straightforward problem which is tackled as before. First we find a particular integral to annihilate the forcing in the differential equation. Recall that

$$\nabla_R^2 \equiv \frac{1}{R^2}\, \frac{\partial}{\partial R}\left(R^2\, \frac{\partial}{\partial R}\right) + \frac{1}{R^2}\, \frac{1}{\sin\theta}\, \frac{\partial}{\partial\theta}\left(\sin\theta\, \frac{\partial}{\partial\theta}\right)$$

and that

$$\frac{1}{\sin\theta}\, \frac{\partial}{\partial\theta}\left(\sin\theta\, \frac{\partial}{\partial\theta}\, P_n(\cos\theta)\right) + n(n+1)\, P_n(\cos\theta) = 0,$$

and then it is not hard to construct a particular integral

$$\frac{1}{6}R - \frac{1}{27}\, \frac{P_2(\cos\theta)}{R}$$

so that we write

$$\Phi_3 = \frac{1}{6}R - \frac{1}{27}\, \frac{P_2(\cos\theta)}{R} + \psi \tag{6.23}$$

Then

$$\left.\begin{aligned} \nabla_R^2\, \psi &= 0 \\ \frac{\partial\psi}{\partial R} &= -\frac{1}{6}\cos^4\theta - \frac{1}{6} - \frac{1}{27}\, P_2(\cos\theta) \text{ on } R = 1 \end{aligned}\right. \tag{6.24}$$

Rewriting $\cos^4\theta$ as

$$\frac{8}{35}\, P_4(\cos\theta) + \frac{4}{7}\, P_2(\cos\theta) + \frac{1}{5}$$

we find

$$\psi = \frac{4}{525}\, \frac{P_4(\cos\theta)}{R^5} + \frac{5}{567}\, \frac{P_2(\cos\theta)}{R^3} + \frac{1}{5R}$$

plus a general inner eigensolution. Since the outer expansion begins with $O(\epsilon^3)$, $\epsilon^4 \Phi_3(r/\epsilon)$ cannot contain any terms larger than ϵ^3 and therefore the most general eigensolution possible at this stage is

$$A_0^{(3)} + A_1^{(3)} P_1(\cos\theta)\left\{R + \frac{1}{2}\frac{1}{R^2}\right\}.$$

Hence

$$\Phi_3 = \frac{1}{6}R - \frac{1}{27}\frac{P_2(\cos\theta)}{R} + \frac{4}{525}\frac{P_4(\cos\theta)}{R^5} + \frac{5}{567}\frac{P_2(\cos\theta)}{R^3} + \frac{1}{5R}$$

$$+ A_0^{(3)} + A_1^{(3)} P_1(\cos\theta)\left\{R + \frac{1}{2}\frac{1}{R^2}\right\} \qquad (6.25)$$

and we now have $\Phi^{(4)}$, that is, the inner expansion through $O(\epsilon^4)$, determined apart from $A_0^{(3)}$ and $A_1^{(3)}$.

Next we calculate $\Phi^{(4,5)}$, and transform it back to the variable R, with the result

$$\Phi^{(4,5)} = \Phi^{(3,5)}(\text{eqn. } 6.18) + \epsilon^4\left\{\frac{1}{6}R - \frac{1}{27}\frac{P_2(\cos\theta)}{R} + \frac{1}{5R}\right.$$

$$\left. + A_0^{(3)} + A_1^{(3)} R\cos\theta\right\} \qquad (6.26)$$

while

$$\phi^{(5,4)} = \phi^{(5,3)}(\text{eqn. } 6.19) + \epsilon^4\frac{R}{6} - \frac{\epsilon^4}{12}\frac{\partial}{\partial X}(R^2) + \epsilon^4\frac{a^{(1)}}{R} - \frac{\epsilon^4}{2}a_{11}^{(1)}\frac{\partial^2}{\partial X^2}(R). \qquad (6.27)$$

Performing the differentiations, using (6.8) for $a_{11}^{(1)}$ and matching (6.26) to (6.27) gives

$$\frac{1}{6}R - \frac{1}{54}\frac{(3\cos^2\theta - 1)}{R} + \frac{1}{5R} + A_0^{(3)} + A_1^{(3)} R\cos\theta$$

$$\equiv \frac{1}{6}R - \frac{1}{6}R\cos\theta + \frac{a^{(1)}}{R} + \frac{1}{18}\left(\frac{1}{R} - \frac{\cos^2\theta}{R}\right)$$

so that

$$\left.\begin{aligned} A_0^{(3)} &= 0 \\ A_1^{(3)} &= -\frac{1}{6} \\ a^{(1)} &= \frac{22}{135} \end{aligned}\right\} \qquad (6.28)$$

Thus when four inner terms and two outer terms are matched these terms are all uniquely determined. Note that in the fourth inner term an eigensolution is present which grows linearly as $R \to \infty$. In still higher approximations the further eigensolutions are also needed in the inner series in order to match the positive powers of R which arise from the expansion of the phase terms $\exp(i\epsilon R)$ in the outer series.

There is no particular merit, nor any great difficulty, in carrying the inner and outer expansions further, and so one can determine the near and far fields to any order in ϵ, though the algebraic complexity increases rapidly with the order. The point we want to demonstrate here is that adherence to the formal procedures provides a straightforward and routine way of carrying the approximations through a number of terms, and that the matching rule can be used to indicate how the expansions should proceed and what form of eigensolution is sufficient at each stage.

7. TWO DIMENSONAL PROBLEMS: LOGARITHMIC GAUGE FUNCTIONS

Logarithmic gauge functions arise in many problems, and the aim of this section is to show how they arise and how they can be handled by MAE in the context of two-dimensional acoustic scattering problems.

We start by looking at the scattering of a plane wave $\phi^i = \exp(i\, k_0 x')$ by the rigid fixed cylinder $r' = L$, so that the problem for the scattered field ϕ' is

$$\left.\begin{aligned} &(\nabla'^2 + k_0^2)\phi' = 0 \\ &\frac{\partial\phi'}{\partial r'} = -\, i\, k_0 \cos\theta \exp(i\, k_0 L \cos\theta) \quad \text{on } r' = L \\ &\phi' \sim (r')^{-1/2} \exp(i\, k_0\, r')\, f(\theta) \quad \text{as } r' \to \infty \end{aligned}\right\} \tag{7.1}$$

the last of these being the two-dimensional radiation condition. In terms of outer and inner coordinates, we have

$$\left.\begin{aligned} &(\nabla_r^2 + 1)\phi = 0 \\ &\frac{\partial\phi}{\partial r} = -\, i \cos\theta \exp(i\epsilon \cos\theta) \quad \text{on } r = \epsilon \\ &\phi \sim r^{-1/2} \exp(ir)\, g(\theta) \quad \text{as } r \to \infty \end{aligned}\right\} \tag{7.2}$$

and

$$\left.\begin{aligned} &(\nabla_R^2 + \epsilon^2)\Phi = 0 \\ &\frac{\partial\Phi}{\partial R} = -\, i\epsilon \cos\theta \exp(i\epsilon \cos\theta) \quad \text{on } R = 1 \\ &\Phi \sim R^{-1/2} \exp(i\epsilon R)\, h(\theta) \quad \text{as } \epsilon R \to \infty \end{aligned}\right\} \tag{7.3}$$

the scattered field being, of course, symmetric about the line $\theta = 0$.

An inner eigensolution $\Lambda_\nu(R)$ of integral order ν is defined as

$$\Lambda_\nu(R) = (R^\nu + R^{-\nu}) \cos \nu\theta \tag{7.4}$$

for $\nu = 0, 1, 2, \ldots$, and satisfies

$$\nabla_R^2 \Lambda_\nu(R) = 0$$

$$\frac{\partial \Lambda_\nu}{\partial R} = 0 \quad \text{on } R = 1 .$$

The general outer eigensolution can be represented in various forms. The commonest representation would be in terms of a sum of Hankel functions of the first kind and of all integral orders,

$$\sum_{\nu=0}^{\infty} a_\nu H_\nu^{(1)}(r) \cos \nu\theta ,$$

the typical term of which varies with r like $r^{-1/2} \exp(ir)$ for large r and so satisfies the radiation condition. Here we stick to our previous type of representation in terms of multipole derivatives of the fundamental solution $H_0^{(1)}(r)$ of the Helmholtz equation in two dimensions. Thus we write the general outer eigensolution as

$$aH_0^{(1)}(r) + a_i \frac{\partial}{\partial x_i} H_0^{(1)}(r) + a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} H_0^{(1)}(r) + \ldots \tag{7.5}$$

(x_1, x_2) being the position coordinates in a plane perpendicular to the axis of the cylinder. As before, the solution symmetric about the x_1-axis, $\theta = 0$, is of the form

$$a\, H_0^{(1)}(r) + a_1 \frac{\partial}{\partial x} H_0^{(1)}(r) + a_{11} \frac{\partial^2}{\partial x^2} H_0^{(1)}(r) + \ldots \tag{7.6}$$

An advantage of using the form (7.6) is that we need only to know the behaviour of the zeroth order Hankel function as $r \to 0$, which is

$$H_o^{(1)}(r) \sim \left[1 - \frac{1}{4} r^2 + 0(r^4)\right]$$

$$+ \frac{2i}{\pi} (\ell n\, r + \gamma_E - \ell n\, 2) \left(1 - \frac{1}{4} r^2 + 0(r^4)\right) + \frac{2i}{\pi} \left(\frac{1}{4} r^2 + 0(r^4)\right) \tag{7.7}$$

where $\gamma_E = 0.5772 \ldots$ is the Euler constant.

As in the previous sections, the first inner problem determines how the expansions start off. Motivated by the boundary condition in (7.3) we assume

$$\Phi \sim \epsilon\, \Phi_o + \ldots$$

and then

$$\nabla_R^2\, \Phi_o = 0$$

$$\frac{\partial \Phi_o}{\partial R} = - i \cos\theta \quad \text{on } R = 1$$

so that

$$\Phi_o = \frac{i \cos\theta}{R} + \sum_{\nu = o}^{\infty} A_\nu^{(o)}\, \Lambda_\nu(R). \tag{7.8}$$

Then

$$\Phi^{(1)}\left(\frac{r}{\epsilon}\right) = \epsilon\, \Phi_o \left(\frac{r}{\epsilon}\right) = \frac{i \epsilon^2 \cos\theta}{r} + \sum_{\nu = o}^{\infty} \epsilon\, A_\nu^{(1)} \left\{\left(\frac{r}{\epsilon}\right)^\nu + \left(\frac{\epsilon}{r}\right)^\nu\right\} \cos \nu\, \theta\, ,$$

and *unless all the* $A_\nu^{(o)}$ *are zero*, the leading term of the expansion of $\Phi^{(1)}(r/\epsilon)$ will contain terms which are either constant or which grow with r. Such terms cannot be matched to the leading order solution, because that is an eigensolution like (7.6) whose *leading order* terms all contain either $\ell n\, r$ or inverse powers of r. Therefore we take

$$\Phi^{(1)}\left(\frac{r}{\epsilon}\right) = \frac{i \epsilon^2 \cos\theta}{r} \tag{7.9}$$

which shows that the leading order term in the outer expansion must be $0(\epsilon^2)$, so that

$$\phi \sim \epsilon^2 \phi_o + \ldots \tag{7.10}$$

where ϕ_o is given by (7.6) with superscript $^{(o)}$ on the coefficients.

Because the leading inner term is $O(\epsilon)$, $\epsilon^2\phi_o(\epsilon R)$ cannot be larger than $O(\epsilon)$, and so

$$\phi_o = a^{(o)}H_o^{(1)}(r) + a_1^{(o)} \frac{\partial}{\partial x} H_o^{(1)}(r) \tag{7.11}$$

is the most general possibility, consisting of an acoustic monopole and an acoustic dipole respectively. Then

$$\epsilon^2\phi_o(\epsilon R) = a^{(o)}\epsilon^2 \frac{2i}{\pi}\left(\ell n\, R + \ell n \epsilon + \gamma_E - \ell n\, 2 - \frac{\pi i}{2}\right)$$

$$+ O(\epsilon^4 \ell n \epsilon, \epsilon^4) + \epsilon\, a_1^{(o)} \frac{2i}{\pi} \frac{\cos\theta}{R} + O(\epsilon^3 \ell n \epsilon, \epsilon^3) \tag{7.12}$$

which gives

$$\phi^{(2,1)} = \epsilon\, a_1^{(o)} \frac{2i}{\pi} \frac{\cos\theta}{R} \tag{7.13}$$

Applying the matching rule

$$\phi^{(2,1)} = \Phi^{(1,2)}$$

gives

$$a_1^{(o)} = \frac{\pi}{2} \tag{7.14}$$

The situation is precisely as in the spherical problem. First order matching determines the dipole coefficient, but does not determine the monopole part of the first order outer field. To find that we have to take careful account of small phase variations in the scattering region which are responsible for an acoustically efficient source mechanism. Analytically, we expand (7.12) to next order to see how the inner series should progress and find that the next terms are $O(\epsilon^2 \ell n \epsilon)$ and $O(\epsilon^2)$, just as indicated in (7.12). We therefore assume

$$\Phi \sim \epsilon\, \Phi_o + \epsilon^2 \ell n \epsilon\, \Phi_1 + \epsilon^2 \Phi_2 + \ldots \tag{7.15}$$

and we shall assume the matching rule

$$\phi^{(2,2)} = \epsilon^2 a^{(o)} \frac{2i}{\pi}\left(\ell n \epsilon + \ell n\, R + \gamma_E - \ell n\, 2 - \frac{\pi i}{2}\right) + \frac{i\epsilon}{\pi R}\cos\theta = \Phi^{(2,2)} \tag{7.16}$$

where we understand that $\epsilon^2\, \ell n \epsilon\, \Phi_1$ and $\epsilon^2\Phi_2$ are to be taken as a single term $O(\epsilon^2)$.

Φ_1 is of course an inner eigensolution, and $\epsilon^2\Phi_1(r/\epsilon)$ can be no larger than $O(\epsilon^2)$, since that is the order of the outer potential. Thus

$$\Phi_1 = A_o^{(1)} \tag{7.17}$$

is the only possibility. For Φ_2 we have the problem

$$\left.\begin{aligned} \nabla_R^2 \Phi_2 &= 0 \\ \frac{\partial \Phi_2}{\partial R} &= \cos^2\theta \quad \text{on } R = 1 \end{aligned}\right\} \tag{7.18}$$

This problem introduces the two-dimensional source potential $\ell n\, R$. Writing the boundary condition as

$$\frac{\partial \Phi_2}{\partial R} = \frac{1}{2}(1 + \cos 2\theta)$$

we look for a solution

$$\Phi_2 = \alpha\, \ell n\, R + \beta \frac{\cos 2\theta}{R^2}$$

we find $\alpha = 1/2$, $\beta = -1/4$, so that

$$\Phi_2 = \frac{1}{2}\ell n\, R - \frac{1}{4}\frac{\cos 2\theta}{R^2} + A_o^{(2)}, \tag{7.19}$$

the form of the eigensolution being determined by the same argument as for (7.17). Thus we have

$$\Phi^{(2)}(R) = \frac{i\epsilon \cos\theta}{R} + \epsilon^2 \ell n\epsilon\, A_o^{(1)} + \epsilon^2 \left\{ \frac{1}{2}\ell n\, R - \frac{1}{4}\frac{\cos 2\theta}{R^2} + A_o^{(2)} \right\},$$

$$\begin{aligned} \Phi^{(2)}\left(\frac{r}{\epsilon}\right) = {} & \frac{i\epsilon^2 \cos\theta}{r} + \epsilon^2 \ell n\epsilon\, A_o^{(1)} + \epsilon^2 \left\{ \frac{1}{2}\ell n\, r - \frac{1}{2}\ell n\epsilon \right\} \\ & - \frac{1}{4}\epsilon^4 \frac{\cos 2\theta}{r^2} + \epsilon^2 A_o^{(2)} \end{aligned} \tag{7.20}$$

which gives

$$\Phi^{(2,2)} = \frac{i\epsilon^2 \cos\theta}{r} + \epsilon^2 \ell n\epsilon \left(A_o^{(1)} - \frac{1}{2} \right) + \epsilon^2 \frac{1}{2}\ell n\, r + \epsilon^2 A_o^{(2)} \tag{7.21}$$

Matching (7.21) to (7.16) gives

$$\left.\begin{aligned} A_o^{(1)} &= \frac{1}{2} \\ a^{(o)} &= \frac{\pi}{4i} \\ A_o^{(2)} &= \frac{1}{2}\left(\gamma_E - \ell n\, 2 - \frac{\pi i}{2}\right) \end{aligned}\right\} \qquad (7.22)$$

which yields the monopole strength $a^{(o)}$ and *non-zero* eigenfunctions in the second and third inner terms Φ_1 and Φ_2.

If we now go back to $\Phi^{(2)}(r/\epsilon)$, as given by (7.20), we see that since $A_o^{(1)} = 1/2$ it contains terms $O(\epsilon^2)$ which are matched to the outer solution, and then continues with an $O(\epsilon^4)$ quadrupole term $\cos 2\theta/r^2$. This suggests that we continue the outer series in the form

$$\phi \sim \epsilon^2 \phi_o + \epsilon^4 \phi_2 + \ldots \qquad (7.23)$$

The choice of ϕ_2 here, rather than ϕ_1, is deliberate. ϕ_2 is again an outer eigensolution, and as in the sphere problem we can allow up to three derivatives of $H_o^{(1)}(r)$ in ϕ_2, but not more, in order that $\epsilon^4 \phi_2(\epsilon R)$ be no larger, as $\epsilon \to o$, than the leading inner term, which is $O(\epsilon)$. Thus

$$\phi_2 = a^{(2)} H_o^{(1)}(r) + a_1^{(2)} \frac{\partial}{\partial x} H_o^{(1)}(r) + a_{11}^{(2)} \frac{\partial^2}{\partial x^2} H_o^{(1)}(r) + a_{111}^{(2)} \frac{\partial^3}{\partial x^3} H_o^{(1)}(r), \qquad (7.24)$$

and without taking the inner solution further, the only matching rule we can use is that

$$\Phi^{(2,4)}(\text{eqn. } 7.20) = \phi^{(4,2)}.$$

The right hand side of this equation can be calculated directly, with some effort, from the assumed form (7.24) and the expansion (7.7). One finds then that

$$\left.\begin{aligned} a_{111}^{(2)} &= 0, \\ -\frac{2i}{\pi} a_{11}^{(2)} &= -\frac{1}{4} \end{aligned}\right\} \qquad (7.25)$$

and the monopole and dipole coefficients remain undetermined.

Expansion of $\phi^{(4)} \equiv (\epsilon^2 \phi_o + \epsilon^4 \phi_2)(\epsilon R)$ beyond $O(\epsilon^2)$ produces terms $O(\epsilon^3 \ell n \epsilon)$ and $O(\epsilon^3)$, so that the inner expansion must take the form

$$\Phi \sim \epsilon\, \Phi_o + \epsilon^2 \ell n \epsilon\, \Phi_1 + \epsilon^2 \Phi_2 + \epsilon^3 \ell n \epsilon\, \Phi_3 + \epsilon^3 \Phi_4 + \ldots \qquad (7.26)$$

The solutions for Φ_3 and Φ_4 are

$$\Phi_3 = A_o^{(3)} + A_1^{(3)}(R + R^{-1}) \cos\theta$$

$$\Phi_4 = A_o^{(4)} + A_1^{(4)}(R + R^{-1}) \cos\theta$$

$$- \frac{i}{2} R \,\ell n\, R \cos\theta - \frac{i}{24} \frac{\cos 3\theta}{R^3} - \frac{7i}{8} \frac{\cos\theta}{R} \tag{7.27}$$

and the matching rule to be applied next is

$$\Phi^{(3,4)} = \phi^{(4,3)} \tag{7.28}$$

Most of the details of this matching are unimportant. One aspect serves, however, as a vital warning. We see from (7.27) that in $\Phi^{(3,4)}$ there will be a term

$$- \frac{i}{2} \epsilon^4 r \,(\ell n\, r - \ell n \epsilon) \cos\theta \tag{7.29}$$

in terms of outer variables, and to this order there are no other inner terms of this form. In the outer expansion (7.23) there are also no terms of the required form; certainly there are terms which contain $\ell n \epsilon$ *when expressed in inner variables,* but when these terms are reexpressed in outer variables the $\ln \epsilon$ disappears from them. Therefore the outer expansion as it stands contains no term which can possibly match (7.29) according to the rule (7.28) - and (7.29) has a definite non-zero coefficient, because the term in (7.27) which gives rise to it, namely $- i/2 \; R \,\ell n\, R \cos\theta$, is a particular integral for the equation

$$\nabla_R^2 \Phi_4 = - \Phi_1 = - \frac{i \cos\theta}{R} .$$

The fault lies with the assumed form (7.23) for the outer expansion. Although the expansion of $\Phi^{(2)}(r/\epsilon)$ produced only algebraic terms $0(\epsilon^2, \epsilon^4)$, indicating the form (7.23) *the presence of $\ell n \epsilon$ terms in the inner expansion at one stage must be taken as a warning that they will probably arise at the next outer stage.* Thus in place of (7.23) we should have anticipated

$$\phi \sim \epsilon^2 \phi_o + \epsilon^4 \,\ell n \epsilon\, \phi_1 + \epsilon^4 \phi_2 + \ldots \tag{7.30}$$

and this choice enables all the matchings to be effected without difficulty - though the algebraic details are horrendous. A further point to be noted is that because ϕ_1 necessarily contains Hankel functions which have logarithmic singularities, the expansion of $\epsilon^4 \,\ell n \epsilon\, \phi_1$

will involve a term $\epsilon^4 \ln^2\epsilon$ – and therefore the inner expansion presumably takes the form

$$\Phi \sim \epsilon\,\Phi_0 + \epsilon^2 \ln\epsilon\,\Phi_1 + \epsilon^2\Phi_2 + \epsilon^3 \ln\epsilon\,\Phi_3 + \epsilon^3\Phi_4$$

$$+ \epsilon^4 \ln^2\epsilon\,\Phi_5 + \epsilon^4 \ln\epsilon\,\Phi_6 + \epsilon^4\Phi_7 + \ldots \qquad (7.31)$$

It is necessary in fact to determine all the eight functions in (7.31) in order to uniquely determine the three functions in the outer wave field (7.30).

Once again then, low frequency acoustic problems bring out subleties in the matching procedure. Although we have not proved it here, it can be shown that failure to regard $\epsilon^3 \ln\epsilon\,\Phi_3$ and $\epsilon^3\Phi_4$ in (7.26) as a single "block" leads to an incorrect determination of some of the coefficients in the outer terms ϕ_1 and ϕ_2 (see [9] and [7]). We have also seen how the presence of logarithmic terms may be indicated by systematic use of the matching principle, and that once they have arisen their presence must be suspected in all subsequent terms of both expansions.

8. PURELY LOGARITHMIC GAUGE FUNCTIONS: SCATTERING BY SOFT BODIES

The problem of plane wave scattering by a soft cylinder – on which the total potential is zero – provides an excellent illustration of the way in which purely logarithmic gauge functions may arise, leading to series which are useless for practical purposes because of the slowness of their convergence. We show how naive application of the matching principle fails, but then show how modification of the gauge functions together with an *insistence* on all matchings enables the slowly convergent series to be "renormalised" into a rapidly convergent series.

If the cylinder is circular, of radius L, then the inner and outer problems for the scattered field ϕ corresponding to an incident potential $\exp(ik_0 x')$ are, respectively,

$$\left.\begin{aligned} (\nabla_R^2 + \epsilon^2)\,\Phi &= 0 \\ \Phi = -\exp(i\epsilon\cos\theta) &\quad \text{on } R = 1 \end{aligned}\right\} \qquad (8.1)$$

$$\left.\begin{aligned} (\nabla_r^2 + 1)\,\phi &= 0 \\ \phi \sim r^{-1/2}\exp(ir)\,f(\theta) &\quad \text{as } r \to \infty \end{aligned}\right\} \qquad (8.2)$$

The inner eigensolutions this time satisfy

$$\nabla_R^2 \Phi = 0, \quad \Phi = 0 \quad \text{on } R = 1 \quad \text{and are}$$

$$\left.\begin{aligned} \Lambda_o &= \ln R \\ \Lambda_\nu &= (R^\nu - R^{-\nu}) \cos \nu\theta \quad (\nu = 1, 2, \ldots) \end{aligned}\right\} \tag{8.3}$$

while the outer eigensolutions are, as before, of the general form

$$a\, H_o^{(1)}(r) + a_1 \frac{\partial}{\partial x} H_o^{(1)}(r) + a_{11} \frac{\partial^2}{\partial x^2} H_o^{(1)}(r) + \ldots \tag{8.4}$$

The boundary condition in (8.1) indicates that $\Phi = 0(1)$ as $\epsilon \to o$, so that if

$$\left.\begin{aligned} &\Phi \sim \Phi_o + \ldots \\ &\text{then} \\ &\Phi_o = -1 + \sum_{\nu=o}^{\infty} A_\nu^{(o)} \Lambda_\nu (R) \end{aligned}\right\} \tag{8.5}$$

and we cannot decide which eigensolutions may be permitted without looking next at the outer field. Suppose that ϕ starts off with an algebraic gauge function $\phi \sim \epsilon^\alpha \phi_o$ with $\alpha > 0$ (on physical grounds). Then the leading term of $\epsilon^\alpha \phi_o(\epsilon R)$ must be $0(1)$ as $\epsilon \to o$ in order to match Φ_o. The only possibility is that α is an integer and that the first term permitted in (8.4) for ϕ_o contains α derivatives. But then that term would contain a factor $R^{-\alpha}$, whereas the leading term in the expansion of $\Phi_o(r/\epsilon)$ contains only r^β or $\ln r$ where $\beta \geqslant 0$. Matching to leading order is therefore impossible if ϕ starts off with an algebraic gauge function. A little trial and error, based on the fact that a monopole $a^{(o)} H_o^{(1)}(r)$ is certain to be present in ϕ_o, suggests that instead we should try

$$\phi \sim \frac{1}{\ln \epsilon} \phi_o + \ldots \tag{8.6}$$

Then *only* the monopole can be present in ϕ_o, for the presence of a dipole, $a_1\, \partial/\partial x\, H_o^{(1)}(r)$ would make $(1/\ln\epsilon)\, \phi_o(\epsilon R)$ infinite like $(\epsilon \ln\epsilon)^{-1}$, rather than $0(1)$ which it needs to be. Thus

$$\phi_o = a^{(o)} H_o^{(1)}(r) \tag{8.7}$$

and with an obvious extension of our previous notation,

$$\phi^{\left(\frac{1}{\ln\epsilon}\right)}(\epsilon R) = \frac{1}{\ln\epsilon}\, a^{(o)}\, H_o^{(1)}(\epsilon R) \tag{8.8}$$

$$\sim \frac{a^{(o)}}{\ln\epsilon}\,\frac{2i}{\pi}\ln\epsilon \quad \text{to leading order.}$$

Hoping that we can extend the matching principle to cover this sort of case we write

$$\phi^{\left(\frac{1}{\ln\epsilon},\, o\right)} = \frac{2i}{\pi}\, a^{(o)}$$

and try matching this to the terms up to $0(1/\ln\epsilon)$ in

$$\Phi^{(o)}\left(\frac{r}{\epsilon}\right) = -1 + A_o^{(o)}(\ln r - \ln\epsilon) + \sum_{\nu=1}^{\infty} A_\nu^{(o)}\left\{\left(\frac{r}{\epsilon}\right)^{\nu} - \left(\frac{\epsilon}{r}\right)^{\nu}\right\}\cos\nu\theta .$$

This will only be possible if all the $A_\nu^{(o)} \equiv 0$ and then

$$\Phi^{\left(0,\, \frac{1}{\ln\epsilon}\right)} = -1 ,$$

which gives us

$$a^{(o)} = +\frac{\pi i}{2} \tag{8.9}$$

This determines the leading order solutions as

$$\Phi \sim -1, \quad \phi \sim \frac{1}{\ln\epsilon}\,\frac{\pi i}{2}\, H_o^{(1)}(r) \tag{8.10}$$

To improve upon these approximations we have to expand $(1/\ln\epsilon)\,\phi_o(\epsilon R)$ up beyond the $0(1)$ term, for expansion of $\Phi_o = -1$ is not helpful. This gives

$$\phi^{\left(\frac{1}{\ln\epsilon}\right)}(\epsilon R) = -1 - \frac{1}{\ln\epsilon}\left\{\ln R + \gamma_E - \ln 2 - \frac{\pi i}{2}\right\} + 0\left(\frac{\epsilon^2}{\ln\epsilon}\right)$$

and suggests

$$\Phi \sim -1 + \frac{1}{\ln\epsilon}\,\Phi_1 + \ldots \tag{8.11}$$

Φ_1 is an inner eigensolution, and some thought shows that the algebraic type of eigensolution cannot yet enter into the solutions. (Note that our previous arguments as to how each expansion determines the form of the other have to some extent broken down with purely logarithmic gauge functions; for example, the outer expansion of $\Phi_o = -1$ is -1, but this does not mean here that the outer series start with an $O(1)$ term). We try

$$\Phi_1 = A_o^{(1)} \ln R \tag{8.12}$$

and then we have

$$\Phi^{\left(\frac{1}{\ln\epsilon}\right)} = -1 + \frac{1}{\ln\epsilon} A_o^{(1)} \ln R,$$

$$\Phi^{\left(\frac{1}{\ln\epsilon}, \frac{1}{\ln\epsilon}\right)} = -1 - A_o^{(1)} + \frac{A_o^{(1)}}{\ln\epsilon} \ln r \tag{8.13}$$

On the other hand,

$$\phi^{\left(\frac{1}{\ln\epsilon}, \frac{1}{\ln\epsilon}\right)} = -1 - \frac{1}{\ln\epsilon}\left(\ln R + \gamma_E - \ln 2 - \frac{\pi i}{2}\right)$$

$$= -\frac{\ln r}{\ln\epsilon} - \frac{1}{\ln\epsilon}\left(\gamma_E - \ln 2 - \frac{\pi i}{2}\right) \tag{8.14}$$

Clearly, (8.13) and (8.14) do not match, though the choice $A_o^{(1)} = -1$ does at least match the term in $(\ln r)/(\ln\epsilon)$. We can either be satisfied with the matching that does work, and which determines $A_o^{(1)}$ uniquely, and merely ignore the failure of other terms to match, or we can think about trying to modify the expansions a little in order that matching of all terms can take place satisfactorily.

This difficulty often arises in the case of purely logarithmic gauge functions. Fraenkel [13] has discussed it at some length, showing how the matching failure can be traced to at best "marginal overlap" of the inner and outer expansions when inverse powers of $\ln\epsilon$ are used as gauge functions. He also shows how, in principle, the terms which cannot be expected to match, can be traced at each stage. It should, therefore, be possible to persist with our expansions in inverse powers of $\ln\epsilon$, and to correctly determine all terms, regardless of the matching failure, by careful attention to the points made by Fraenkel. But the result would be useless for practical purposes, for the coefficients would all be of order one and an ϵ of 10^{-10} would still only make consecutive terms smaller by a factor of 10 or so.

We recommend the unrelenting insistence on the matching rule, combined with a slight change to the gauge functions, as a systematic and effective way of redeeming the situation on all counts. As a first step, we try taking $(\ln\epsilon + K)^{-1}$ as the basic gauge function, in place of $(\ln\epsilon)^{-1}$, K being a constant. Assume

$$\left.\begin{aligned} \phi &\sim \frac{1}{(\ell n \epsilon + K)} \bar{\phi}_o + \ldots \\ \Phi &\sim -1 + \frac{1}{(\ell n \epsilon + K)} \bar{\Phi}_1 + \ldots \end{aligned}\right\} \tag{8.15}$$

and as before, appropriate choices for the eigenfunctions are

$$\left.\begin{aligned} \bar{\phi}_o &= \bar{a}^{(o)} H_o^{(1)}(r), \\ \bar{\Phi}_1 &= \bar{A}_o^{(1)} \ell n R . \end{aligned}\right\} \tag{8.16}$$

Now,

$$\bar{\phi}\left(\frac{1}{\ell n \epsilon + K}\right)(\epsilon R) = \frac{\bar{a}^{(o)}}{(\ell n \epsilon + K)} \left\{1 + \frac{2i}{\pi}(\ell n R + \ell n \epsilon + K + \gamma_E - \ell n 2 - K) + \ldots\right\}$$

so that

$$\bar{\phi}\left(\frac{1}{\ell n \epsilon + K}, \frac{1}{\ell n \epsilon + K}\right) = + \frac{\bar{a}^{(o)}}{(\ell n \epsilon + K)} \left[1 + \frac{2i}{\pi}(\ell n r + \gamma_E - \ell n 2)\right] \tag{8.17}$$

while

$$\bar{\Phi}\left(\frac{1}{\ell n \epsilon + K}, \frac{1}{\ell n \epsilon + K}\right) = -1 - \bar{A}_o^{(1)} + \frac{(K + \ell n r)\bar{A}_o^{(1)}}{(\ell n \epsilon + K)} \tag{8.18}$$

Equations (8.17) and (8.18) can now be matched, and give

$$\left.\begin{aligned} \bar{a}^{(o)} &= + \frac{\pi i}{2} \\ \bar{A}_o^{(1)} &= -1 \end{aligned}\right\} \tag{8.19}$$

as before, while the remaining terms can be matched if we choose

$$K = \gamma_E - \ell n 2 - \frac{\pi i}{2} . \tag{8.20}$$

Now we have the solutions

$$\left.\begin{aligned} \phi &\sim \frac{1}{(\ell n \epsilon + K)} \cdot \frac{\pi i}{2} H_o^{(1)}(r), \\ \Phi &\sim -1 - \frac{\ell n R}{(\ell n \epsilon + K)} . \end{aligned}\right\} \tag{8.21}$$

and to take the expansions further we expand $\phi(\epsilon R)$ beyond $(\ell n\epsilon + K)^{-1}$. We find

$$\phi(\epsilon R) \sim -1 - \frac{\ell n\, R}{(\ell n\epsilon + K)} + 0\left(\frac{\epsilon^2}{\ell n\epsilon + K}, \epsilon^2\right)$$

which indicates an inner expansion

$$\Phi \sim -1 - \frac{\ell n\, R}{(\ell n\epsilon + K)} + \frac{\epsilon^2}{(\ell n\epsilon + K)}\Phi_2 + \epsilon^2\Phi_3 + \ldots \tag{8.22}$$

This shows the value of enforcing the matching rules in terms of slightly strained gauge functions. Instead of having the slow series in inverse powers of $\ell n\epsilon$, we have effectively now summed that series, or at any rate a subset of that series, into the term $(\ell n\epsilon + K)$, and the next term, $0(\epsilon^2)$, is now very much smaller than $1/(\ell n\epsilon)^2$, as it would otherwise have been. Note, of course, that now we are out of the infinite sequence of purely logarithmic functions we must expect to have to treat

$$\frac{\epsilon^2}{(\ell n\epsilon + K)}\Phi_2 \quad \text{and} \quad \epsilon^2\Phi_3$$

together as effectively a single $0(\epsilon^2)$ term in the matching.

Failure of matching at higher orders in this and similar problems indicates that a further straining of the gauge functions is necessary; the overlap has become marginal, or perhaps has disappeared altogether, at higher order in terms of powers of ϵ and inverse powers of $(\ell n\epsilon + K)$. The remedy may be to try

$$\ell n\epsilon + K \rightarrow \ell n\epsilon + K + K_1\epsilon$$

or more generally $\ell n\epsilon + K \rightarrow \ell n\epsilon + K + K_1(\epsilon)$ where $K_1(\epsilon) \rightarrow o$ as $\epsilon \rightarrow o$ and is to be determined by the enforcement of all matchings.

9. COMPOSITE EXPANSIONS

Occasionally it is useful to be able to combine the inner and outer expansions into a single smooth approximation, valid everywhere from the scattering body to infinity in the scattering type of problem, for example. (More often, however, one needs just to know the directivity pattern of the field at infinity, or the pressure on the scatterer, and these can be obtained directly from the outer expansion alone, or the inner expansion alone, respectively.) There are two common ways of combining the inner and outer series - the methods of ADDITIVE and MULTIPLICATIVE COMPOSITION. In the additive method it is customary to add $\phi^{(m)}(r)$ say to $\Phi^{(n)}(R)$ and take off the part $\phi^{(m,n)}$ (or $\Phi^{(n,m)}$, they are identical) which expansions $\phi^{(m)}$ and $\Phi^{(n)}$ have in common so that it is not counted twice. This gives a composite

$$\phi_{mn}^{AC} \equiv \phi^{(m)} + \Phi^{(n)} - \phi^{(m,n)} \tag{9.1}$$

which is often claimed to be as good as the inner or outer series in their respective domains, and so, if the expansions overlap, to provide a single smooth function equivalent in accuracy to the two separate expansions. In the method of multiplicative composition one defines

$$\phi_{mn}^{MC} \equiv \frac{\phi^{(m)} \cdot \Phi^{(n)}}{\phi^{(m,n)}} \tag{9.2}$$

which again is claimed to provide a different, though asymptotically equivalent, single smooth function, good through $0(\epsilon^m)$ in the outer region and through $0(\epsilon^n)$ in the inner.

As Van Dyke points out in the Notes to the revised edition of his book, these two composites are particular instances of a very general class of composites. Let F be a general functional, F^{-1} its inverse. Then a general composite formed from the outer expansion $\phi^{(m)}$ through $0(\epsilon^m)$ and the inner expansion $\Phi^{(n)}$ through $0(\epsilon^n)$ is defined by

$$\phi_{mn}^{F} = F^{-1}\left\{F(\phi^{(m)}) + F(\Phi^{(n)}) - F(\phi^{(m,n)})\right\} \tag{9.3}$$

The functional $F(\phi) \equiv \phi$ gives the additive composition, the functional $F(\phi) \equiv \ell n\, \phi$ gives the multiplicative rule. Other functionals F give different composites, as when $F(\phi) = \exp \phi$ for example, which gives

$$\phi_{mn} = \ell n \left\{\exp \phi^{(m)} + \exp \Phi^{(n)} - \exp \phi^{(m,n)}\right\} \tag{9.4}$$

Now in general *none* of these composites has the accuracy claimed for it. This follows from the fact that in general neither $\phi^{(m,n)}$ nor $\Phi^{(n,m)}$ has any asymptotic significance with respect to ϕ or Φ, as stated already on pages 11 and 12 in connection with the Asymptotic Matching Principle. The only thing that can be said about the additive or multiplicative composites is that if, for example, ϕ_{mn}^{AC} $(r, R = r/\epsilon, \epsilon)$ is expanded for fixed r and $\epsilon \to o$, and if that expansion is truncated beyond $0(\epsilon^m)$ the result will be precisely $\phi^{(m)}$, the outer expansion through $0(\epsilon^m)$. Likewise if ϕ_{mn}^{AC} $(r = \epsilon R, R, \epsilon)$ is expanded through $0(\epsilon^n)$ for fixed R, the result will be the inner expansion $\Phi^{(n)}$ through $0(\epsilon^n)$ - and the same is true of the multiplicative composite. The same is *not* true, however, for any functional F other than the additive one ($F \equiv 1$) or the multiplicative one ($F \equiv \ell n$) because of the presence of extra terms which will be thrown up when one has to expand not simply $\Phi^{(n)}(r/\epsilon)$, for example, but $F(\Phi^{(n)}(r/\epsilon))$.

Therefore a composite is at best only as good as $\phi^{(m)}$ or $\Phi^{(n)}$ *when it has been expanded and truncated*, and otherwise may not have the accuracy of $\phi^{(m)}$ in the outer region or of $\Phi^{(n)}$ in the inner region, while its accuracy or otherwise in between - say for $r = 0(\epsilon^{1/2})$ - is in general quite unknown. This last point is emphatically made by Schneider [14] who gives an example in which the multiplicative composite is hopeless in the intermediate range of r because of a *zero* in the denominator, $\phi^{(m,n)}$. Van Dyke [1, Notes to revised edition] gives examples in which different functionals F can be used to produce widely differing accuracies of the composites formed according to (9.3).

My feeling at the moment then is that there is no known rule which invariably produces a composite as good as the inner or outer expansions taken separately, let alone in between, and therefore that composites are to be avoided wherever possible. In simple cases, and often to leading order only, it may be possible to form a composite by careful inspection of the behaviour of the inner and outer expansions in their own domains and in the overlap domain. If that can be done, the additive rule should be used, as the most fundamental, though the multiplicative rule usually gives much neater results.

Just to show how these composites are formed, according to the usual, though generally incorrect prescription, we go back to the field scattered by a compact rigid sphere, for which

$$\begin{aligned} \phi &\sim \epsilon^3 \left\{ -\frac{1}{3}\frac{e^{ir}}{r} - \frac{i}{2}\frac{\partial}{\partial x}\frac{e^{ir}}{r} \right\} = \phi^{(3)} \\ \Phi &\sim \epsilon \frac{i}{2R^2} \cos\theta = \Phi^{(1)} \end{aligned} \tag{9.5}$$

to leading order. We have

$$\phi^{(3,1)} = \Phi^{(1,3)} = \epsilon \frac{i}{2R^2} \cos\theta = \Phi^{(1)},$$

so that additive and multiplicative composition both produce the same composite which is actually $\phi^{(3)}$ itself. In other words

$$\epsilon^3 \left\{ -\frac{1}{3}\frac{e^{ir}}{r} - \frac{i}{2}\frac{\partial}{\partial x}\frac{e^{ir}}{r} \right\}$$

is a uniformly valid description of the whole field, to leading order. However, the accuracy of this leading order approximation is *not* uniform, as is only to be expected in a singular perturbation problem. In the outer field the next term is $O(\epsilon^5)$ while $\phi^{(3)}$ is $O(\epsilon^3)$, whereas in the inner field $\phi^{(3)}$ is $O(\epsilon)$ and the next term is $O(\epsilon^2)$.

To improve accuracy in the near field we can try using

$$\phi \sim \epsilon^3 \left\{ -\frac{1}{3} \frac{e^{ir}}{r} - \frac{i}{2} \frac{\partial}{\partial x} \frac{e^{ir}}{r} \right\} = \phi^{(3)}$$

with

$$\Phi \sim \epsilon \frac{i}{2R^2} \cos\theta - \epsilon^2 \left\{ \frac{2}{9} \frac{P_2(\cos\theta)}{R^3} + \frac{1}{3R} \right\} = \Phi^{(2)} ,$$

and

$$\phi^{(3,2)} = \Phi^{(2,3)} = -\epsilon^2 \frac{1}{3R} + \epsilon \frac{i}{2R^2} \cos\theta .$$

Then the additive (3, 2) composite is

$$\epsilon^3 \left\{ -\frac{1}{3} \frac{e^{ir}}{r} - \frac{i}{2} \frac{\partial}{\partial x} \frac{e^{ir}}{r} \right\} - \epsilon^2 \left\{ \frac{2}{9} \frac{P_2(\cos\theta)}{R^3} \right\}$$

and for once this is neater than the multiplicative composite. No significance is to be attached to the last term here in the wave zone $r = O(1)$, for there it is $O(\epsilon^5)$ and the outer expansion itself is only good to $O(\epsilon^3)$ so far. The inclusion of the last term when $r = O(1)$ may make a distinct difference to the value of ϕ for moderately small values of ϵ, and may apparently improve upon the asymptotic representation $\phi^{(3)}$. In general, however, such an improvement is coincidental, and the ϕ_{32} composite has no asymptotic correctness until it is expanded for fixed r and terms smaller than $O(\epsilon^3)$ thrown away, which of course takes us back to the outer expansion $\phi^{(3)}$ alone.

10. CONCLUSIONS

These notes have attempted to show MAE at work on some simple, but hopefully representative, problems of classical linear acoustics. Despite their apparent physical and mathematical simplicity, these low frequency sound generation and scattering problems illustrate very effectively a number of subleties which have often gone unnoticed in more complicated, but perhaps less delicate, problems of fluid dynamics. They show how *careful* application of the method leads in a reasonably straightforward way to solutions in the near and far fields whose accuracy is limited only by the length of the algebraic manipulations that have to be carried out at high order.

Modern acoustics, and expecially underwater acoustics, deals with many problems involving one or more small parameters. For example, the fluid is generally only weakly compressible, so that often the Helmholtz numbers $k_0 L$ may be *small;* a vibrating surface coupled to an acoustic fluid may be subject to *high* or *low* fluid loading; the acoustic fluid may be in bulk motion at a *low* Mach number; the phase speed of an elastic surface wave may be only *slightly less* than the sonic speed, or the frequency may be *very much less* than the coincidence frequency, etc., etc. All such problems can be regarded as governed by two distinct length scales, typically an acoustic length scale and the other scale determined by hydrodynamic or elastic effects regardless of compressibility. As such, we must expect them to be singular perturbation problems, and must look to MAE as a likely method of attack. A variety of problems in both linear and non-linear acoustics is discussed from this viewpoint in the article [9], while still more recent and unpublished work will deal with MAE in problems involving coupled wave-bearing media.

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